

A.E. CONVERGENCE AND 2-WEIGHT INEQUALITIES FOR POISSON-LAGUERRE SEMIGROUPS

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ABSTRACT. We find optimal decay estimates for the Poisson kernels associated with various Laguerre-type operators L . From these, we solve two problems about the Poisson semigroup $e^{-t\sqrt{L}}$. First, we find the largest space of initial data f so that $e^{-t\sqrt{L}}f(x) \rightarrow f(x)$ at *a.e. x*. Secondly, we characterize the largest class of weights w which admit 2-weight inequalities of the form $\|\sup_{0 < t \leq t_0} |e^{-t\sqrt{L}}f|\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, for some other weight v .

1. INTRODUCTION

In this paper we continue the research, started in [6, 5], about Poisson integrals associated with certain differential operators L , say symmetric and positive in $L^2(\emptyset, \mu)$. Namely, we are interested in the behavior of

$$u(t, x) = e^{-t\sqrt{L}}f(x)$$

as a solution of the elliptic differential equation

$$\begin{cases} u_{tt} - Lu = 0 \\ u(0, x) = f(x), \end{cases} \quad \text{in the half-plane } (0, \infty) \times \emptyset.$$

We shall study two questions, which are closely related among themselves

- (i) find the **largest** class of functions f for which $\lim_{t \rightarrow 0^+} u(t, x) = f(x)$, *a.e. x* $\in \emptyset$;
- (ii) establish 2-weight inequalities of the form

$$\left\| \sup_{0 < t \leq t_0} |u(t, x)| \right\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$$

for the **largest** class of weights w for which a suitable v with this property exists.

In [5] we considered these questions in full detail when L is the Hermite operator. In this paper we intend to do the same for the various *Laguerre operators*. We remark that solving these questions for the **largest** class of weights or initial data is generally not an easy task, requiring **optimal** decay estimates of the Poisson kernels. These new estimates have an independent interest and may be useful in other settings; see e.g. recent work by Liu and Sjögren [7]. For our purposes, they will provide a.e. convergence for new initial data f compared to Muckenhoupt [8] and Stempak [12], and also for larger weight classes compared to those of Nowak in [9].

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We now state our results, for simplicity in the special case of the classical Laguerre operator in $\mathbb{R}_+ := (0, \infty)$

$$(1.1) \quad \mathbb{L} = -y \partial_{yy} - (\alpha + 1 - y) \partial_y + m, \quad \text{where } \alpha > -1 \quad \text{and} \quad m \geq 0.$$

We have incorporated a parameter $m \geq 0$ which later will allow us to recover other Laguerre-type operators (after suitable changes of variables). We shall also consider a slightly more general family of partial differential equations, namely

$$(1.2) \quad \begin{cases} u_{tt} + \frac{1-2\nu}{t} u_t = \mathbb{L}u \\ u(t, 0) = f, \end{cases} \quad \text{where } \nu > 0.$$

These pde's appear in relation with the *fractional operator* $f \mapsto \mathbb{L}^\nu f$ (see e.g. [14]).

As discussed in [14], a candidate solution to (1.2) is given by the *Poisson-like integral*

$$(1.3) \quad P_t f(x) := \frac{t^{2\nu}}{4^\nu \Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u}} [e^{-u\mathbb{L}} f](x) \frac{du}{u^{1+\nu}}, \quad t > 0,$$

which is subordinated to the “heat” semigroup $\{e^{-u\mathbb{L}}\}_{u>0}$. Our first goal is to find the most general conditions on a function $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ so that $P_t f$ is a meaningful solution of (1.2). These conditions will depend on the following weight

$$(1.4) \quad \Phi(y) = \frac{y^\alpha e^{-y}}{(1+y)^m [\ln(e+y)]^{1+\nu}} \quad \text{if } m > 0, \quad \text{and} \quad \Phi(y) = \frac{y^\alpha e^{-y}}{[\ln(e+y)]^\nu} \quad \text{if } m = 0.$$

Theorem 1.1. *For every $f \in L^1(\Phi)$ the function $u(t, x) = P_t f(x)$ in (1.3) is defined by an absolutely convergent integral such that*

- (i) $u(t, x) \in C^\infty((0, \infty) \times \mathbb{R}_+)$ and satisfies the pde (1.2)
- (ii) $\lim_{t \rightarrow 0^+} u(t, x) = f(x)$ at a.e. $x \in \mathbb{R}_+$.

Conversely, if a function $f \geq 0$ is such that the integral in (1.3) is finite for some $(t, x) \in (0, \infty) \times \mathbb{R}_+$, then f must necessarily belong to $L^1(\Phi)$.

We recall that Muckenhoupt proved in [8] the pointwise convergence for data f in the smaller space $L^1(y^\alpha e^{-y} dy)$ (in the classical setting, i.e., (1.1) with $m = 0$ and (1.3) with $\nu = 1/2$). Our result, which is sharp on positive f , enlarges the space to

$$L^1(y^\alpha e^{-y} / \sqrt{\log(e+y)} dy),$$

and allows to include new initial data such as $f(y) = e^y / [(1+y)^{\alpha+1} \log(e+y)]$.

Our second question concerns more “quantitative” bounds for the solutions of (1.2), expressed in terms of the following *local maximal operators*

$$(1.5) \quad P_{t_0}^* f(x) := \sup_{0 < t \leq t_0} |P_t f(x)|, \quad \text{with } t_0 > 0 \text{ fixed.}$$

From our estimates of the Poisson kernels we shall be able to prove that

$$P_{t_0}^* : L^1(\Phi) \rightarrow L_{\text{loc}}^s \quad \text{if } s < 1, \quad \text{and} \quad P_{t_0}^* : L^1(\Phi) \cap L_{\text{loc}}^p \rightarrow L_{\text{loc}}^p \quad \text{if } p > 1.$$

However, our main interest is to obtain global bounds in x , which we shall phrase through the following problem.

Problem 1. A 2-weight problem for the operator $P_{t_0}^*$. *Given $1 < p < \infty$, characterize the class of weights $w(x) > 0$ such that $P_{t_0}^*$ maps $L^p(w) \rightarrow L^p(v)$ boundedly, for some other weight $v(x) > 0$.*

Our second main result gives a complete answer to Problem 1. For $p \in (1, \infty)$ we define the class of weights

$$(1.6) \quad D_p(\Phi) = \left\{ w(y) > 0 : \|w^{-\frac{1}{p}} \Phi\|_{L^{p'}(\mathbb{R}_+)} < \infty \right\}.$$

Observe that $L^p(w) \subset L^1(\Phi)$ if and only if $w \in D_p(\Phi)$, so in view of Theorem 1.1, this is a necessary condition for Problem 1. Our second theorem shows that it is also sufficient.

Theorem 1.2. *Let $1 < p < \infty$ and $t_0 > 0$ be fixed. Then, for a weight $w(x) > 0$ the condition $w \in D_p(\Phi)$ is equivalent to the existence of some other weight $v(x) > 0$ such that*

$$(1.7) \quad P_{t_0}^* : L^p(w) \rightarrow L^p(v) \quad \text{boundedly.}$$

Moreover, for every $\varepsilon > 0$, we can choose a weight $v \in D_{p+\varepsilon}(\Phi)$ satisfying (1.7).

We remark that Problem 1 is only a “one-side” problem, in contrast with the (more difficult) question of characterizing all pairs of weights (w, v) for which (1.7) holds. One-side problems were considered in the early 80s by Rubio de Francia [11] and Carleson and Jones [1] for various classical operators. Here we shall follow the approach by the latter, which has the advantage of giving **explicit** expressions for the second weight $v(x)$ (see Remark 6.2 below). This is also a novelty compared to [5], where we used the non-constructive method of Rubio de Francia.

We can now briefly describe our approach to the proofs in this paper. Rather than working with the operator \mathbb{L} , most of our computations will involve the “squared” Laguerre operator

$$(1.8) \quad L = -\partial_{yy} + \left[y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2} \right] + 2\mu.$$

This has the advantage of resembling the Hermite operator (which is the case $\alpha = -1/2$), so at some points we may use computations from [5]. There are however various additional difficulties which are characteristic of the Laguerre setting. The term $1/y^2$ produces a singularity when $y \rightarrow 0$ which must be handled separately from the singularity at $y \rightarrow \infty$. This is reflected in the behavior of the Bessel function I_α which is part of the kernel expression of e^{-uL} . One may also expect additional difficulties when $\alpha \in (-1, -1/2)$ (cases

sometimes avoided in the literature, but that we consider here), related to the fact that such Laguerre functions blow-up when $y \rightarrow 0$.

Most of our work will be employed in deriving *precise decay estimates* for the Poisson kernel, which will lead to the following control of the operator $P_{t_0}^*$

$$(1.9) \quad P_{t_0}^* f(x) \lesssim C(x) \left[\mathcal{M}^{\text{loc}}(f\Phi)(x) + \int_{\mathbb{R}_+} f\Phi \right],$$

for a reasonably well-behaved $C(x)$ (to be absorbed later as part of the weight $v(x)$). Here \mathcal{M}^{loc} denotes a *local* Hardy-Littlewood maximal operator in \mathbb{R}_+ , given by

$$(1.10) \quad \mathcal{M}^{\text{loc}} f(x) := \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I(x,r)} |f(y)| \chi_{\{\frac{x}{2} \leq y \leq Mx\}} dy$$

for a suitable $M > 1$. We also use the notation $I(x,r) = (x-r, x+r) \cap \mathbb{R}_+$.

Finally, we remark that the statement of Theorems 1.1 and 1.2 remains true when \mathbb{L} is replaced by any of the Laguerre-type operators in the table below, provided Φ in (1.4) is replaced by the corresponding function in the table. As in (1.4), in the extremal case $\mu = -(\alpha+1)$, the logarithmic term in the denominator of Φ must be replaced by $[\log(e+y)]^\nu$.

Eigenfunctions	differential operator	function Φ
L_n^α	$\mathbb{L} = -y\partial_{yy} - (\alpha+1-y)\partial_y + m \quad \left. \vphantom{\mathbb{L}} \right\} \quad m \geq 0$	$\frac{y^\alpha e^{-y}}{(1+y)^m [\ln(e+y)]^{1+\nu}}$
φ_n^α	$L = -\partial_{yy} + y^2 + \frac{1}{y^2}(\alpha^2 - \frac{1}{4}) + 2\mu \quad \left. \vphantom{L} \right\} \quad \mu \geq -(\alpha+1)$	$\frac{y^{\alpha+\frac{1}{2}} e^{-y^2/2}}{(1+y)^{1+\alpha+\mu} [\ln(e+y)]^{1+\nu}}$
\mathfrak{L}_n^α	$\mathfrak{L} = -y\partial_{yy} - \partial_y + \frac{1}{4}\left[y + \frac{\alpha^2}{y}\right] + \frac{\mu}{2} \quad \left. \vphantom{\mathfrak{L}} \right\} \quad \mu \geq -(\alpha+1)$	$\frac{y^{\frac{\alpha}{2}} e^{-y/2}}{(1+y)^{\frac{1+\alpha+\mu}{2}} [\ln(e+y)]^{1+\nu}}$
ℓ_n^α	$\mathcal{L} = -y\partial_{yy} - (\alpha+1)\partial_y + \frac{y}{4} + \frac{\mu}{2} \quad \left. \vphantom{\mathcal{L}} \right\} \quad \mu \geq -(\alpha+1)$	$\frac{y^\alpha e^{-y/2}}{(1+y)^{\frac{1+\alpha+\mu}{2}} [\ln(e+y)]^{1+\nu}}$
ψ_n^α	$\Lambda = -\partial_{yy} - \frac{2\alpha+1}{y}\partial_y + y^2 + 2\mu \quad \left. \vphantom{\Lambda} \right\} \quad \mu \geq -(\alpha+1)$	$\frac{y^{2\alpha+1} e^{-y^2/2}}{(1+y)^{1+\alpha+\mu} [\ln(e+y)]^{1+\nu}}$

TABLE 1. Table of Φ -functions for various Laguerre-type operators.

The outline of the paper will be the following. In §2 we consider a version of Theorem 1.1 for *heat integrals* $u(t, x) = e^{-tL}f(x)$, which are solutions of the heat equation

$$u_t + Lu = 0 \quad \text{in } (0, T) \times \mathbb{R}_+, \quad \text{with } u(0, x) = f(x).$$

Heat integrals are easier to handle, and the explicit expression of the heat kernel, $e^{-tL}(x, y)$, makes more transparent the behavior we shall later encounter in Poisson kernels. In §3 we study 2-weight inequalities for the local maximal operator \mathcal{M}^{loc} . In §4 we apply these to prove a version of Theorem 1.2 for heat integrals. In §5 we take up the study of Poisson integrals, splitting in various subsections the detailed kernel estimates leading to (1.9). In §6 we shall give the proof of Theorems 1.1 and 1.2 for the operator L . Finally, in §7 we show how to transfer the results to the Laguerre operators in Table 1.

Throughout the paper $\alpha > -1$ is fixed, as are the parameters μ, m in the differential operators. The notation $A \lesssim B$ will mean $A \leq cB$, for a constant $c > 0$ which may depend on α, μ and other parameters like p, M, t_0, ε , but not on t, x, y . If needed, we shall stress the latter dependence by $c(x), c(t, x), \dots$. Finally, if $1 < p < \infty$ we set $p' = p/(p-1)$.

2. THE SIMPLER MODEL OF HEAT INTEGRALS

In this section L will denote the Laguerre-type operator

$$(2.1) \quad L = -\partial_{yy} + \left[y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2} \right],$$

that is, we have set $\mu = 0$ in (1.8)*. The corresponding eigenfunctions $\{\varphi_n^\alpha\}_{n=0}^\infty$ satisfy

$$L\varphi_n^\alpha = (4n + 2\alpha + 2)\varphi_n^\alpha, \quad n = 0, 1, 2, \dots$$

and form an orthonormal basis of $L^2(0, \infty)$. The kernel of the associated heat semigroup e^{-tL} , written in terms of the new variable $s = \text{th } t$, has the explicit expression

$$(2.2) \quad \begin{aligned} e^{-tL}(x, y) &= \sum_{n=0}^{\infty} e^{-(4n+2\alpha+2)t} \varphi_n^\alpha(x) \varphi_n^\alpha(y) \\ &= \sqrt{\frac{1-s^2}{2s}} \bar{I}_\alpha\left(\frac{(1-s^2)xy}{2s}\right) e^{-\frac{(x-y)^2}{4s}} e^{-\frac{s(x+y)^2}{4}}. \end{aligned}$$

Here we have used the convenient notation $\bar{I}_\alpha(z) = \sqrt{z}e^{-z}I_\alpha(z)$, so that $\bar{I}_\alpha(z) \approx \langle z \rangle^{\alpha+\frac{1}{2}}$, with $lz = \min\{z, 1\}$.

2.1. A.e. convergence of heat integrals. We wish to establish the pointwise convergence of $e^{-tL}f(x)$ with the weakest possible conditions in f . For this purpose, the following kernel bound will suffice

$$(2.3) \quad e^{-tL}(x, y) \lesssim \left\langle \frac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} \frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{s}}.$$

*For heat integrals this implies no loss, since the general case can be factored as $e^{-2\mu t}[e^{-tL}f](x)$.

To produce this bound from (2.2) one disregards the last exponential, and uses $1 - s^2 \leq 1$ when $\alpha \geq -\frac{1}{2}$. If $\alpha \in (-1, -\frac{1}{2})$, note that $l\lambda z \geq \lambda z$ for $\lambda \leq 1$, so one can leave outside a power $(1 - s^2)^{\alpha+1} \leq 1$.

Theorem 2.1. *Let $\alpha > -1$ be fixed, and f be such that*

$$(2.4) \quad \int_0^\infty |f(y)| e^{-ay^2} l y^{\alpha+\frac{1}{2}} dy < \infty, \quad \text{for some (possibly large) } a > 0.$$

Then,

$$\lim_{t \rightarrow 0^+} e^{-tL} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}_+.$$

PROOF: For each fixed $N \geq 2$ it suffices to show that $\lim_{t \rightarrow 0^+} e^{-tL} f(x) = f(x)$ for a.e. $x \in (1/N, N)$. We split

$$f = f\chi_{\{0 < y \leq 2N\}} + f\chi_{\{y > 2N\}} = f_1 + f_2.$$

The function f_1 has bounded support and belongs to $L^1(y^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2}} dy)$, so we can apply the results of Muckenhoupt [8] (with a suitable change of variables[†], as indicated by Stempak [12]) to obtain

$$\lim_{t \rightarrow 0^+} e^{-tL} f_1(x) = f_1(x) = f(x), \quad \text{a.e. } x \in [\frac{1}{N}, N].$$

Next we shall show that, under the hypothesis (2.4),

$$\lim_{t \rightarrow 0^+} e^{-tL} f_2(x) = 0, \quad \forall x \in [\frac{1}{N}, N].$$

Since $t \rightarrow 0$, we may assume that $s = \text{th } t \leq s_0$ for some $s_0 < \frac{1}{10}$ (which we shall make precise below). Note that $\frac{1}{N} \leq x \leq N$ and $y > 2N$ imply that

$$\left\langle \frac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} = 1, \quad \forall s < 1.$$

So, by (2.3), in this region we have a gaussian bound for the kernel

$$e^{-tL}(x, y) \lesssim s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}} \leq s^{-\frac{1}{2}} e^{-\frac{y^2}{16s}},$$

using in the last step that $|x - y| \geq y/2$. Choosing $s_0 < \frac{1}{32a}$ (with a as in (2.4)), we see that for all $y > 2N$,

$$e^{-tL}(x, y) \lesssim s^{-\frac{1}{2}} e^{-\frac{y^2}{32s}} e^{-ay^2} \leq s^{-\frac{1}{2}} e^{-\frac{N^2}{8s}} e^{-ay^2}$$

and therefore

$$e^{-tL} f_2(x) \lesssim s^{-\frac{1}{2}} e^{-\frac{N^2}{8s}} \int_{y>2N} |f(y)| e^{-ay^2} dy \longrightarrow 0, \quad \text{as } s \rightarrow 0^+.$$

□

[†]See e.g. §§7.2 and 7.4 below for the explicit change of variables.

2.2. Heat kernel estimates. The estimates in the proof of Theorem 2.1, slightly refined in some steps, lead to the following proposition.

Proposition 2.2. *Let $\alpha > -1$. Then, for every $\gamma > 1$ there is some $M = M_\gamma > 1$ such that*

$$(2.5) \quad e^{-tL}(x, y) \leq C_\gamma \begin{cases} \frac{e^{-\frac{|x-y|^2}{4s}}}{\sqrt{s}} \left\langle \frac{xy}{s} \right\rangle^{\alpha+\frac{1}{2}} & \text{if } \frac{x}{2} \leq y \leq Mx \\ c(x) \langle y \rangle^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2\gamma \text{th}(2t)}} & \text{if } y < \frac{x}{2} \text{ or } y > Mx \end{cases},$$

for all $x, y \in \mathbb{R}_+$ and $s = \text{th } t \in (0, 1)$. Here we can set $c(x) = 1/lx^{\alpha+\frac{3}{2}}$.

PROOF: Given x, y and s , for simplicity we write $z = \frac{xy}{s}$. Our estimates below will follow from

$$(2.6) \quad e^{-tL}(x, y) \lesssim \langle z \rangle^{\alpha+\frac{1}{2}} s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}} e^{-\frac{sy^2}{4}}.$$

This clearly implies the estimate in the local part $y \in [\frac{x}{2}, Mx]$, so we shall look at the complementary range. Below we shall ignore the last exponential factor in (2.6), and observe that all our estimates will end up with $e^{-y^2/(4\gamma s)}$. Combining these two one obtains the asserted exponential bound, since

$$e^{-\frac{y^2}{4\gamma s}} e^{-\frac{sy^2}{4}} \leq e^{-\frac{y^2}{4\gamma}(s+\frac{1}{s})} = e^{-\frac{y^2}{2\gamma \text{th}(2t)}},$$

as $s + s^{-1} = 2/\text{th}(2t)$ when $s = \text{th } t$.

To handle the kernel expression in (2.6) we need to separate the cases $z \leq 1$ and $z \geq 1$. We begin with $z \geq 1$. In the region $y > Mx$ we may use $|x - y| \geq (1 - \frac{1}{M})y$ to obtain

$$e^{-tL}(x, y) \lesssim \frac{e^{-(\frac{M-1}{M})^2 \frac{y^2}{4s}}}{\sqrt{s}} \leq c_M \frac{e^{-(\frac{M-1}{M})^3 \frac{y^2}{4s}}}{y} \leq c_M e^{-\frac{y^2}{4s\gamma}} \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{lx^{\alpha+\frac{3}{2}}},$$

where in the last step we select $M = M_\gamma$ sufficiently large so that $(\frac{M}{M-1})^3 \leq \gamma$, and have used the trivial estimate

$$(2.7) \quad \frac{1}{y} \leq \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{ly^{\alpha+\frac{3}{2}}} \leq \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{lx^{\alpha+\frac{3}{2}}}, \quad \text{if } y \geq x.$$

On the other hand, if $y < x/2$ we have $|x - y| \geq x/2$, which leads to

$$e^{-tL}(x, y) \lesssim \frac{e^{-\frac{(x/2)^2}{4s}}}{\sqrt{s}} \leq c_\gamma \frac{e^{-\frac{(x/2)^2}{4s\gamma}}}{x} \leq c_\gamma \frac{e^{-\frac{y^2}{4\gamma s}}}{x}.$$

In the case $\alpha \in (-1, -\frac{1}{2})$ this can be combined with

$$(2.8) \quad \frac{1}{x} \leq \frac{\langle x \rangle^{\alpha+\frac{1}{2}}}{lx^{\alpha+\frac{3}{2}}} \leq \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{lx^{\alpha+\frac{3}{2}}}, \quad \text{since } x \geq y.$$

If on the contrary $\alpha \geq -\frac{1}{2}$, we can insert $1 \leq z^{\alpha+\frac{1}{2}}$ in the gaussian bound and obtain

$$(2.9) \quad e^{-tL}(x, y) \lesssim \frac{e^{-\frac{(x/2)^2}{4s}}}{\sqrt{s}} \left(\frac{xy}{s}\right)^{\alpha+\frac{1}{2}} = \left(\frac{x^2}{s}\right)^{\alpha+1} e^{-\frac{(x/2)^2}{4s}} \frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}} \leq c_\gamma e^{-\frac{y^2}{4s\gamma}} \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{lx^{\alpha+\frac{3}{2}}}.$$

This completes the proof of (2.5) when $z \geq 1$.

We turn to the case $z \leq 1$, and replace the gaussian bound by

$$(2.10) \quad e^{-tL}(x, y) \lesssim s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4s}} z^{\alpha+\frac{1}{2}}.$$

This is a better bound when $\alpha \geq -\frac{1}{2}$, so some of the previous arguments also lead to (2.5); namely one can disregard z in the region $y > Mx$, and must keep it when $y < \frac{x}{2}$ and argue as in (2.9). We are left with the case $\alpha \in (-1, -\frac{1}{2})$, which makes $z^{\alpha+\frac{1}{2}} \geq 1$. In the region $y > Mx$, this can be absorbed by the exponentials since

$$\begin{aligned} e^{-tL}(x, y) &\lesssim \left(\frac{xy}{s}\right)^{\alpha+\frac{1}{2}} s^{-\frac{1}{2}} e^{-\frac{(M-1)^2}{4s}} \frac{y^2}{4s} = \left(\frac{y^2}{s}\right)^{\alpha+1} e^{-\frac{(M-1)^2}{4s}} \frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \\ &\leq c_M e^{-\frac{(M-1)^3}{4s}} \frac{y^2}{4s} \frac{\langle x \rangle^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}}}{ly^{2\alpha+2}} \leq c_M e^{-\frac{y^2}{4s\gamma}} \frac{\langle x \rangle^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}}}{lx^{2\alpha+2}}, \end{aligned}$$

using in the last step that $y \geq x$. Finally, in the region $y < \frac{x}{2}$ the inequalities in (2.9) remain also valid, so we have completed the proof of Proposition 2.2. \square

Proposition 2.2 can be expressed in terms of the local Hardy-Littlewood maximal function in \mathbb{R}_+

$$(2.11) \quad \mathcal{M}_M^{\text{loc}} f(x) := \sup_{r>0} \frac{1}{|I(x, r)|} \int_{I_r(x)} |f(y)| \chi_{\{\frac{x}{2} < y < Mx\}} dy,$$

where $I(x, r) = I_r(x)$ denotes the interval $(x-r, x+r) \cap \mathbb{R}_+$.

Corollary 2.3. *Let $\alpha > -1$ and $\gamma > 1$. Then there is some $M = M_\gamma > 1$ such that*

$$(2.12) \quad \sup_{0 < t \leq t_0} |e^{-tL} f(x)| \lesssim \mathcal{M}_M^{\text{loc}} f(x) + c(x) \int_{\mathbb{R}_+} |f(y)| \langle y \rangle^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2\gamma \text{th}(2t_0)}} dy,$$

for every $x, y, t_0 > 0$ and $c(x) = 1/lx^{\alpha+\frac{3}{2}}$.

PROOF: We only have to prove the local estimate, and may assume that $\text{supp } f \subset [\frac{x}{2}, Mx]$. If $s = \text{th } t \leq x^2$, then $z = \frac{xy}{s} \gtrsim 1$ (since $x \approx y$ when $y \in \text{supp } f$), so the gaussian bound of the kernel and a standard slicing argument easily lead to

$$|e^{-tL} f(x)| \lesssim \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} e^{-\frac{(x-y)^2}{4s}} |f(y)| \chi_{\{\frac{x}{2} \leq y \leq Mx\}} dy \lesssim \mathcal{M}_M^{\text{loc}} f(x).$$

If $s = \text{th } t \geq x^2$, then using $x \approx y$,

$$|e^{-tL} f(x)| \lesssim \frac{x^{2\alpha+1}}{s^{\alpha+1}} \int_{\frac{x}{2}}^{Mx} |f(y)| dy \lesssim \frac{x^{2\alpha+1}}{x^{2\alpha+2}} \int_{\frac{x}{2}}^{Mx} |f(y)| dy \lesssim \mathcal{M}_M^{\text{loc}} f(x). \quad \square$$

Remark 2.4. An estimate quite similar to (2.12), with a slightly worse bound for the exponential inside the integral, was obtained by Chicco-Ruiz and Harboure in [2, §5].

3. 2-WEIGHT INEQUALITIES FOR \mathcal{M}^{loc}

This section is about the *local maximal operator* in \mathbb{R}_+

$$\mathcal{M}^{\text{loc}} f(x) := \sup_{t>0} \frac{1}{|I_t(x)|} \int_{I_t(x)} |f(y)| \chi_{\{\frac{x}{2} < y < Mx\}} dy,$$

where $M > 1$ is a fixed parameter. For simplicity, we do not include the subscript M in the notation, but the implicit constants appearing below will all depend on M . For the 1-weight theory of this operator we refer to [10, §6].

For each $p \in (1, \infty)$, consider the following family of weights in \mathbb{R}_+

$$D_p^{\text{loc}} = \left\{ W(x) > 0 : \int_J W^{-\frac{p'}{p}} < \infty, \quad \forall J \Subset (0, \infty) \right\}.$$

Associated with $W \in D_p^{\text{loc}}$, we consider a family of weights $\{V_\varepsilon\}_{\varepsilon>0}$, defined by

$$(3.1) \quad V_\varepsilon(x) = V(x) \rho_\varepsilon[V(x)], \quad \text{where } V(x) := \left[\mathcal{M}^{\text{loc}}(W^{-\frac{p'}{p}})(x) \right]^{-\frac{p}{p'}},$$

and with the notation $\rho_\varepsilon(x) := \min\{x^\varepsilon, x^{-\varepsilon}\}$. Observe that $V_{\varepsilon_2} \leq V_{\varepsilon_1} \leq V \leq W$ if $\varepsilon_1 \leq \varepsilon_2$. This definition is a slight variant of the one proposed by Carleson and Jones in [1], and leads to the following 2-weight inequalities.

Theorem 3.1. *Let $1 < p < \infty$ and $W \in D_p^{\text{loc}}$. Then for every $\varepsilon > 0$*

$$\mathcal{M}^{\text{loc}} : L^p(W) \rightarrow L^p(V_\varepsilon) \quad \text{boundedly,}$$

where V_ε is defined as in (3.1).

PROOF: The argument of the proof is due to Carleson and Jones [1] (see also a recent application in [4, Prop. 4.2]). For completeness, we sketch the modifications required for the local operator \mathcal{M}^{loc} . Call $E_n = \{x \in \mathbb{R}_+ : \mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})(x) < 2^n\}$, $n \in \mathbb{Z}$, and define the operators

$$(3.2) \quad T_n g(x) := \chi_{E_n} \mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}} g)(x).$$

Note that $T_n : L^1(W^{-\frac{1}{p-1}}) \rightarrow L^{1,\infty}$, with a uniform bound in n , since

$$(3.3) \quad \left| \left\{ T_n g(x) > R \right\} \right| \leq \left| \left\{ \mathcal{M}(W^{-\frac{1}{p-1}} g)(x) > R \right\} \right| \leq \frac{c_0}{R} \int_{\mathbb{R}_+} W^{-\frac{1}{p-1}} |g|,$$

using in the last step the weak-1 boundedness of the Hardy-Littlewood maximal operator.

Similarly, $T_n : L^\infty(W^{-\frac{1}{p-1}}) \rightarrow L^\infty$ with $\|T_n\| \leq 2^n$, since

$$(3.4) \quad \|T_n g\|_\infty = \sup_{x \in E_n} |\mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}} g)(x)| \leq 2^n \|g\|_\infty.$$

Thus, by the Marcinkiewicz interpolation theorem we obtain

$$(3.5) \quad \int_{E_n} |T_n(g)|^p \leq c_0 2^{\frac{np}{p'}} \int_{\mathbb{R}_d} |g|^p W^{-\frac{1}{p-1}}, \quad n \in \mathbb{Z}.$$

Setting $g = fW^{\frac{1}{p-1}}$ in the above inequality, this is the same as

$$(3.6) \quad \int_{E_n} |\mathcal{M}^{\text{loc}}(f)|^p \leq c_0 2^{\frac{np}{p'}} \int_{\mathbb{R}_+} |f|^p W, \quad n \in \mathbb{Z}.$$

Now, modulo null sets $\mathbb{R}_+ = \cup_{n \in \mathbb{Z}} [E_n \setminus E_{n-1}]$ (since $0 < \mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})(x) < \infty$ at *a.e.* x), and we have

$$V_\varepsilon(x) \approx 2^{-\frac{np}{p'}} 2^{-\frac{\varepsilon|n|p}{p'}}, \text{ if } x \in E_n \setminus E_{n-1}.$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+} |\mathcal{M}^{\text{loc}} f|^p V_\varepsilon &\lesssim \sum_{n \in \mathbb{Z}} 2^{-\frac{np}{p'}} 2^{-\frac{\varepsilon|n|p}{p'}} \int_{E_n} |\mathcal{M}^{\text{loc}} f|^p \\ &\stackrel{\text{(by (3.6))}}{\lesssim} \left(\sum_{n \in \mathbb{Z}} 2^{-\frac{\varepsilon|n|p}{p'}} \right) \int_{\mathbb{R}^d} |f|^p W, \end{aligned}$$

as we wished to show. □

The weight V_ε inherits some of the integrability behavior of W if ε is sufficiently small. To state this we first define the subclasses

$$\begin{aligned} D_p^0(\beta) &= \left\{ W \in D_p^{\text{loc}} : \int_0^1 W^{-\frac{p'}{p}}(y) l y^{\beta p'} dy < \infty \right\}, \quad \text{for } \beta > -1, \\ D_p^{\text{exp}}(a) &= \left\{ W \in D_p^{\text{loc}} : \int_1^\infty W^{-\frac{p'}{p}}(y) e^{-ay^{2p'}} dy < \infty \right\}, \quad \text{for } a > 0. \end{aligned}$$

Proposition 3.2. *Let $1 < p < \infty$ and $W \in D_p^{\text{loc}}$. Then, for each $\varepsilon > 0$, the weight defined in (3.1) satisfies $V_\varepsilon \in D_q^{\text{loc}}$, for all $q > p + \varepsilon p/p'$. Moreover, we additionally have*

$$(i) \ W \in D_p^0(\beta) \text{ implies } V_\varepsilon \in D_q^0(\beta), \text{ provided } q > p + \varepsilon \frac{p}{p'} \frac{|1+\beta p'|}{1+\beta}$$

$$(ii) \ W \in D_p^{\text{exp}}(a) \text{ implies } V_\varepsilon \in D_q^{\text{exp}}(b), \text{ provided } q > p(1 + \varepsilon)M^2 a/b.$$

PROOF: Observe that

$$(3.7) \quad V_\varepsilon(x)^{-\frac{q'}{q}} = \max_{\pm} [\mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})(x)]^{\frac{p-1}{q-1}(1 \pm \varepsilon)}.$$

The assumption $q > p + \frac{\varepsilon p}{p'}$ implies that $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then, given $J = [a, b] \in \mathbb{R}_+$,

$$\begin{aligned} \int_J [\mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})]^s &\lesssim |J|^{1-s} \|\mathcal{M}(W^{-\frac{1}{p-1}} \chi_{J^*})\|_{L^{1,\infty}}^s \\ &\lesssim c_J \left(\int_{J^*} W^{-\frac{1}{p-1}} \right)^s < \infty, \end{aligned}$$

with $J^* = [a/2, Mb] \in (0, \infty)$. The same applies if we set $s = \frac{(p-1)(1-\varepsilon)}{q-1}$ (which is also < 1), so we deduce from (3.7) that $\int_J V_\varepsilon^{-\frac{1}{q-1}} < \infty$.

We next prove (i), and as before set $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then, denoting $I_j = [2^{-j-1}, 2^{-j}]$, we have

$$\begin{aligned} \int_0^1 [\mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})]^s ly^{\beta q'} dy &\lesssim \sum_{j=0}^{\infty} 2^{-j\beta q'} |I_j|^{1-s} \|\mathcal{M}(W^{-\frac{1}{p-1}} \chi_{I_j^*})\|_{L^{1,\infty}}^s \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\beta q'} 2^{-j(1-s)} \left(\int_{2^{-j-2}}^{2^{-j}} W^{-\frac{1}{p-1}} \right)^s \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j[\beta(q'-p's)+1-s]} \left(\int_0^M W^{-\frac{1}{p-1}}(y) ly^{\beta p'} dy \right)^s. \end{aligned}$$

This is a finite expression provided

$$\beta(q' - p's) + 1 - s > 0,$$

and using the value of $s = \frac{(p-1)(1+\varepsilon)}{q-1}$ and solving for q this is equivalent to

$$q > p + \frac{\varepsilon p(1 + \beta p')}{p'(1 + \beta)}.$$

In order to have $\int_0^1 V_\varepsilon^{-\frac{1}{q-1}} ly^{\beta q'} dy < \infty$ the previous relation must also hold with ε replaced by $-\varepsilon$, so a sufficient condition is

$$q > p + \frac{\varepsilon p|1 + \beta p'|}{p'(1 + \beta)},$$

as we wished to show.

We now prove (ii). Let $\gamma > 1$ (to be precised later), and as before set $I_j = [\gamma^j, \gamma^{j+1}]$ and $s = \frac{(p-1)(1+\varepsilon)}{q-1} < 1$. Then

$$\begin{aligned} \int_1^\infty [\mathcal{M}^{\text{loc}}(W^{-\frac{1}{p-1}})]^s e^{-by^{2q'}} dy &\lesssim \sum_{j=0}^{\infty} e^{-b\gamma^{2j}q'} \gamma^{(1-s)j} \|\mathcal{M}(W^{-\frac{1}{p-1}} \chi_{I_j^*})\|_{L^{1,\infty}}^s \\ &\lesssim \sum_{j=0}^{\infty} e^{-b\gamma^{2j}q'} \gamma^{(1-s)j} \left(\int_{\gamma^j/2}^{M\gamma^{j+1}} W^{-\frac{1}{p-1}} dy \right)^s \\ &\leq \sum_{j=0}^{\infty} \gamma^{(1-s)j} e^{-\gamma^{2j}[bq' - p'aM^2\gamma^2s]} \left(\int_{1/2}^\infty W^{-\frac{1}{p-1}} e^{-p'ay^2} dy \right)^s. \end{aligned}$$

This is now a finite expression provided

$$bq' > p'aM^2\gamma^2s,$$

which using the value of s and solving for q gives

$$q > p(1 + \varepsilon)M^2\gamma^2a/b.$$

Clearly, we can choose a $\gamma > 1$ with this property under the assumption

$$q > p(1 + \varepsilon)M^2a/b.$$

Since this also implies the validity of the estimates with ε replaced by $-\varepsilon$, we may conclude that $V_\varepsilon \in D_q^{\text{exp}}(b)$, as desired.

□

4. 2-WEIGHT INEQUALITIES FOR LOCAL MAXIMAL HEAT OPERATORS

Let L be as in (2.1), and for each $t_0 > 0$, consider

$$h_{t_0}^* f(x) := \sup_{0 < t \leq t_0} |e^{-tL} f(x)|.$$

Given any $T > t_0$, this operator is well defined over functions $f \in L^1(\varphi_T)$, where

$$\varphi_T(y) = \langle y \rangle^{\alpha + \frac{1}{2}} e^{-\frac{y^2}{2\text{th}(2T)}}.$$

We wish to study 2-weight inequalities for $h_{t_0}^*$ over subspaces $L^p(w) \subset L^1(\varphi_T)$. By duality, the class of weights for which such inclusion holds is given by

$$D_p(\varphi_T) := \left\{ w > 0 : \|w^{-\frac{1}{p}} \varphi_T\|_{p'} < \infty \right\}.$$

Here we show that for all such weights the operator $h_{t_0}^*$ satisfies a 2-weight inequality.

Theorem 4.1. *Let $T > t_0 > 0$ and $1 < p < \infty$. Then, for every $w \in D_p(\varphi_T)$ there exists another weight $v(x) > 0$ such that*

$$h_{t_0}^* : L^p(w) \rightarrow L^p(v), \quad \text{boundedly.}$$

Moreover, if $q > p$ and t_0 is sufficiently small, then we can select $v \in D_q(\varphi_T)$.

Remark 4.2. The second weight $v(x)$ will be constructed explicitly; see (4.2), (4.5) and (4.6) below. Observe that v depends on α, p, t_0, T and of course w .

PROOF of Theorem 4.1: The crucial estimate was already given in Corollary 2.3. We shall use it with the parameter $\gamma = \text{th}(2T)/\text{th}(2t_0) > 1$, which produces a suitable $M = M_\gamma > 1$ such that

$$(4.1) \quad h_{t_0}^* f(x) \lesssim \mathcal{M}_M^{\text{loc}} f(x) + c(x) \int_0^\infty |f(y)| \varphi_T(y) dy.$$

The last integral is bounded by $\|f\|_{L^p(w)} \|w^{-\frac{1}{p}} \varphi_T\|_{p'}$, so the second term will be fine for any weight $v(x)$ such that $c(x) = 1/\langle x \rangle^{\alpha + \frac{3}{2}} \in L^p(v)$. For instance we may take

$$(4.2) \quad v_2(x) = \frac{lx^{(\alpha + \frac{3}{2})p-1}}{[\log(e/lx)]^2 (1+x)}$$

which clearly satisfies

$$\int_0^\infty |c(x)|^p v_2(x) dx = \int_0^\infty \frac{dx}{lx[\log(e/lx)]^2 (1+x)^p} < \infty.$$

Further, we claim that $v_2 \in D_q(\varphi_T)$ iff $q > p$. Indeed, since $\varphi_T(x)$ decays exponentially, it suffices to check the integrability for x near 0. Writing $\beta = \alpha + \frac{1}{2}$ so that

$$\varphi_T(x) \approx lx^\beta \quad \text{and} \quad v_2(x) \approx \frac{lx^{(\beta+1)p-1}}{[\log(e/lx)]^2}$$

we easily see that

$$(4.3) \quad \int_0^1 v_2(x)^{-\frac{q'}{q}} l x^{\beta q'} dx \approx \int_0^1 \frac{[\log(e/x)]^{2q'/q}}{x^{1-q'(\beta+1)(1-\frac{p}{q})}} dx < \infty.$$

For the first term in (4.1) we shall use the results in §3. We first note that

$$(4.4) \quad w \in D_p(\varphi_T) \iff w \in D_p^0(\alpha + \frac{1}{2}) \cap D_p^{\text{exp}}(a), \quad \text{with } a = 1/(2 \text{th } 2T),$$

where the weight classes $D_p^0(\beta)$ and $D_p^{\text{exp}}(a)$ were defined just before Proposition 3.2. Then, for every $\varepsilon > 0$ Theorem 3.1 gives

$$\|\mathcal{M}_M^{\text{loc}} f\|_{L^p(v_{1,\varepsilon})} \lesssim \|f\|_{L^p(w)},$$

provided

$$(4.5) \quad v_{1,\varepsilon}(x) = \mathcal{V}(x) \rho_\varepsilon(\mathcal{V}(x)), \quad \text{where } \mathcal{V}(x) = \left[\mathcal{M}_M^{\text{loc}}(w^{\frac{1}{1-p}})(x) \right]^{1-p}$$

(or $v_{1,\varepsilon} = V_\varepsilon$ in the notation of (3.1)). Hence setting

$$(4.6) \quad v(x) = \min\{v_{1,\varepsilon}(x), v_2(x)\}$$

with $v_{1,\varepsilon}$ and v_2 defined as in (4.5) and (4.2), we have proved that $h_{t_0}^* : L^p(w) \rightarrow L^p(v)$.

It remains to verify the last statement in Theorem 4.1. We already know that, for every $q > p$, we have $v_2 \in D_q(\varphi_T)$. Concerning $v_{1,\varepsilon}$, from the equivalence in (4.4) it suffices to prove that $V_\varepsilon \in D_q^0(\alpha + \frac{1}{2}) \cap D_q^{\text{exp}}(a)$ for a sufficiently small ε . The first assertion is immediate from (i) in Proposition 3.2. However, (ii) in the same proposition only gives $V_\varepsilon \in D_\rho^{\text{exp}}(a)$ if $\rho > p(1 + \varepsilon)M^2$, where $M = M_\gamma$ is the parameter obtained in Proposition 2.2 by the rule $(\frac{M}{M-1})^3 = \gamma = \text{th}(2T)/\text{th}(2t_0)$. If we allow both ε and t_0 be sufficiently small (so that M becomes close enough to 1), then we can set $\rho = q$, and hence conclude that $v_{1,\varepsilon} \in D_q(\varphi_T)$ as desired. \square

5. POISSON KERNEL ESTIMATES

In this section we fix $\alpha > -1$ and $\mu \geq -(\alpha + 1)$, and consider the Laguerre-type operator

$$(5.1) \quad L = -\partial_{yy} + \left[y^2 + \frac{\alpha^2 - \frac{1}{4}}{y^2} \right] + 2\mu,$$

whose eigenfunctions $\{\varphi_n^\alpha\}_{n=0}^\infty$ form an orthonormal basis of $L^2(0, \infty)$, and satisfy

$$L\varphi_n^\alpha = (4n + 2(\alpha + 1 + \mu))\varphi_n^\alpha, \quad n = 0, 1, 2, \dots$$

They can be expressed in terms of the (normalized) Laguerre polynomials L_n^α by

$$(5.2) \quad \varphi_n^\alpha(y) = \sqrt{2} y^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2}} L_n^\alpha(y^2),$$

although we shall not use this formula here. The kernel of the associated heat semigroup, e^{-tL} , can be written explicitly in various forms

$$(5.3) \quad \begin{aligned} e^{-tL}(x, y) &= \sum_{n=0}^{\infty} e^{-[4n+2(\alpha+1+\mu)]t} \varphi_n^\alpha(x) \varphi_n^\alpha(y) \\ (r = e^{-2t}) &= r^\mu \sqrt{\frac{2r}{1-r^2}} \bar{I}_\alpha\left(\frac{2rxy}{1-r^2}\right) e^{-\frac{(x-ry)^2}{1-r^2}} e^{\frac{x^2-y^2}{2}} \end{aligned}$$

$$(5.4) \quad (s = \text{th } t) = \left(\frac{1-s}{1+s}\right)^\mu \sqrt{\frac{1-s^2}{2s}} \bar{I}_\alpha\left(\frac{(1-s^2)xy}{2s}\right) e^{-\frac{(x-y)^2}{4s}} e^{-\frac{s(x+y)^2}{4}}$$

where as before we have set $\bar{I}_\alpha(z) = \sqrt{z}e^{-z}I_\alpha(z)$. Thus, using the notation $lz = \min\{z, 1\}$, we shall have $\bar{I}_\alpha(z) \approx \langle z \rangle^{\alpha+\frac{1}{2}}$. Both expressions of the heat kernel will be useful in our later computations. For instance, (5.3) is good when $r \approx 0$, as it isolates correctly the decaying factor $\langle y \rangle^{\alpha+\frac{1}{2}}e^{-y^2/2}$. On the other hand, (5.4) will be useful when $s \approx 0$ (hence $r \approx 1$), since it makes transparent the gaussian behavior of the singularity $s^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4s}}$.

Using the subordination formula in (1.3), the Poisson kernel associated with L becomes

$$(5.5) \quad P_t(x, y) := \frac{t^{2\nu}}{4^\nu \Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u}} [e^{-uL}(x, y)] \frac{du}{u^{1+\nu}}, \quad t > 0.$$

Changing variables $r = e^{-2u}$ (i.e., $u = \frac{1}{2} \ln \frac{1}{r}$) one sees that

$$P_t(x, y) \approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_0^1 e^{-\frac{t^2}{2 \ln \frac{1}{r}}} r^{\mu+\frac{1}{2}} \frac{e^{-\frac{(x-ry)^2}{1-r^2}} \langle \frac{rxy}{1-r^2} \rangle^{\alpha+\frac{1}{2}}}{\sqrt{1-r} (\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r}.$$

We shall consider two regions of integration according to the behavior of $z := \frac{rxy}{1-r^2}$. The regions will be separated by the number

$$r_0(xy) = \begin{cases} \frac{1}{2xy} & , \text{ if } xy \geq 1 \\ 1 - \frac{xy}{2} & , \text{ if } xy \leq 1. \end{cases}$$

Indeed, it is elementary to check that

- (1) If $0 < r \leq r_0(xy)$ then $z \leq 1$.
- (2) If $r_0(xy) \leq r < 1$ then $z \geq 1/2$.

Thus we can write

$$\begin{aligned} P_t(x, y) &\approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \left[\int_0^{r_0(xy)} \cdots \left(\frac{rxy}{1-r^2} \right)^{\alpha+\frac{1}{2}} \frac{dr}{r} + \int_{r_0(xy)}^1 \cdots \frac{dr}{r} \right] \\ &= B_t(x, y) + A_t(x, y). \end{aligned}$$

The next two propositions summarize the estimates we shall need to handle these kernels. We shall make extensive use of the function

$$(5.6) \quad \Phi(y) := \frac{\langle y \rangle^{\alpha+\frac{1}{2}} e^{-y^2/2}}{(1+y)^{\mu+\frac{1}{2}} [\log(y+e)]^{1+\nu}},$$

with the agreement that in the extreme case $\mu = -(\alpha + 1)$ the log in the denominator is just $[\log(y + e)]^\nu$. The first result gives, for fixed t and x , the *optimal decay* of $y \mapsto P_t(x, y)$ in terms of the function $\Phi(y)$.

Proposition 5.1. *Given $t, x > 0$, there exist $c_1(t, x) > 0$ and $c_2(t, x) > 0$ such that*

$$(5.7) \quad c_1(t, x) \Phi(y) \leq P_t(x, y) \leq c_2(t, x) \Phi(y), \quad \forall y \in \mathbb{R}_+.$$

The second result is a refinement of the upper bound in (5.7) with a few advantages: it is uniform in the variable t , it isolates in the “local part” the singularities of the kernel $P_t(x, y)$, and finally provides “reasonable” bounds for the constant’s dependence on x .

Proposition 5.2. *There exists $M > 1$ such that the following holds for all $t, x, y > 0$*

$$(5.8) \quad P_t(x, y) \lesssim C_1(x) \frac{t^{2\nu} e^{-y^2/2}}{(t + |x - y|)^{1+2\nu}} \chi_{\{\frac{x}{2} < y < Mx\}} + C_2(x) (t \vee 1)^{2\nu} \Phi(y),$$

where $C_1(x) = (1 + x)^{2\nu} e^{\frac{x^2}{2}}$ and $C_2(x) = [\log(e + x)]^{1+\nu} (1 + x)^{|\mu + \frac{1}{2}|} e^{\frac{x^2}{2}} / \langle x \rangle^{\alpha + \frac{3}{2}}$.

If we consider, for fixed $M > 1$, the *local maximal function* in \mathbb{R}_+

$$(5.9) \quad \mathcal{M}_M^{\text{loc}} f(x) := \sup_{t>0} \frac{1}{|I_t(x)|} \int_{I_t(x)} |f(y)| \chi_{\{\frac{x}{2} < y < Mx\}} dy,$$

then we may express (5.8) as follows.

Corollary 5.3. *Let $t_0 > 0$ be fixed. Then there is some $M > 1$ such that*

$$(5.10) \quad P_{t_0}^* f(x) \lesssim C_1(x) \mathcal{M}_M^{\text{loc}}(f e^{-\frac{y^2}{2}})(x) + C_2(x) \|f\|_{L^1(\Phi)}, \quad x \in \mathbb{R}_+,$$

with $C_1(x)$ and $C_2(x)$ as in Proposition 5.2.

This is the key estimate from which we shall deduce the theorems claimed in §1. The interested reader may wish to skip the technical proofs of the propositions in the next subsections and pass directly to §6 for the proof of the theorems.

5.1. Estimates from below for $B_t(x, y)$. Recall that

$$(5.11) \quad B_t(x, y) \approx t^{2\nu} (xy)^{\alpha + \frac{1}{2}} e^{\frac{x^2 - y^2}{2}} \int_0^{r_0(xy)} \frac{r^{\alpha + \mu + 1}}{(1 - r)^{\alpha + 1} (\ln \frac{1}{r})^{1+\nu}} e^{-\frac{t^2}{2 \ln \frac{1}{r}}} e^{-\frac{(x - ry)^2}{1 - r^2}} \frac{dr}{r}.$$

The lower bound in Proposition 5.1 will be obtained by just looking at this integral.

Lemma 5.4. *For fixed $t, x > 0$ it holds*

$$B_t(x, y) \geq c_1(t, x) \Phi(y), \quad \forall y \in \mathbb{R}_+,$$

for a suitable function $c_1(t, x) > 0$.

PROOF: We first look at $y < x \wedge \frac{1}{x}$. Then $xy < 1$, and hence $r_0(xy) > 1/2$. So, we can estimate $B_t(x, y)$ by an integral over $0 < r < \frac{1}{2}$, which disregarding irrelevant terms becomes

$$B_t(x, y) \gtrsim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{\frac{x^2-y^2}{2}} \int_0^{\frac{1}{2}} \frac{r^{\alpha+\mu+1}}{(\ln \frac{1}{r})^{1+\nu}} e^{-\frac{t^2}{2 \ln \frac{1}{r}}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r}.$$

We can get rid of the first two exponentials using

$$e^{\frac{x^2-y^2}{2}} \geq 1 \quad (\text{since } y \leq x) \quad \text{and} \quad e^{-\frac{t^2}{2 \ln \frac{1}{r}}} \geq e^{-\frac{t^2}{2 \ln 2}} \quad (\text{since } r \leq \frac{1}{2}).$$

For the last exponential notice that $0 < x - ry \leq x$, and hence $e^{-\frac{(x-ry)^2}{1-r^2}} \geq e^{-\frac{4}{3}x^2}$. This leaves a convergent integral in r , so we conclude that

$$B_t(x, y) \gtrsim c_1(t, x) \langle y \rangle^{\alpha+\frac{1}{2}},$$

with $c_1(t, x) = t^{2\nu} x^{\alpha+\frac{1}{2}} e^{-\frac{t^2}{2 \ln 2}} e^{-\frac{4}{3}x^2}$. Notice that $y \leq 1$ in this range, so we find the required expression for $\Phi(y)$.

Suppose now that $y \geq x \vee \frac{1}{x}$. Then $xy \geq 1$, and hence $r_0(xy) = \frac{1}{2xy} \leq 1/2$. Arguing as before we can estimate $B_t(x, y)$ by

$$B_t(x, y) \gtrsim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{y^2}{2}} e^{-\frac{t^2}{2 \ln 2}} \int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{(\ln \frac{1}{r})^{1+\nu}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r}.$$

This time we get rid of the exponential inside the integral using

$$|x - ry| \leq x + ry \leq x + \frac{1}{2x} \quad (\text{since } r \leq \frac{1}{2xy}),$$

which implies $e^{-\frac{(x-ry)^2}{1-r^2}} \geq e^{-\frac{4}{3}(x+\frac{1}{x})^2}$. We can easily compute the integral

$$\int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r} \approx \frac{1}{(xy)^{\alpha+\mu+1} [\log 2xy]^{1+\nu}}, \quad \text{if } \alpha + \mu + 1 > 0,$$

with the right hand side becoming $1/[\log 2xy]^\nu$ in the extreme case $\alpha + 1 + \mu = 0$. Since $y > \max\{x, 1\}$, note that

$$\log(2xy) \leq \log(2y^2) \approx \log(y + e).$$

Thus, combining all the previous estimates we conclude that

$$B_t(x, y) \gtrsim c_1(t, x) \frac{e^{-\frac{y^2}{2}}}{y^{\mu+\frac{1}{2}} [\log(y + e)]^{1+\nu}},$$

which, since $y \geq 1$, is the required expression for $\Phi(y)$ (with the usual agreement when $\mu + \alpha + 1 = 0$). In this part we have set $c_1(t, x) = t^{2\nu} e^{-\frac{t^2}{2 \ln 2}} x^{-(\mu+\frac{1}{2})} e^{-\frac{4}{3}(x+\frac{1}{x})^2}$.

Finally, since the function $y \mapsto P_t(x, y)/\Phi(y)$ is continuous and positive, it is also bounded from below by some $c_1(t, x)$ when y belongs to the compact set $[x \wedge \frac{1}{x}, x \vee \frac{1}{x}]$. \square

5.2. Estimates from above for $B_t(x, y)$. The next lemma, combined with the previous one, shows that for fixed t and x , the function $B_t(x, y)$ essentially behaves like $\Phi(y)$.

Lemma 5.5. *For fixed $t, x > 0$ it holds*

$$(5.12) \quad B_t(x, y) \lesssim c(x) \max\{t^{2\nu}, 1\} \Phi(y), \quad \forall y \in \mathbb{R}_+,$$

with $c(x) = 1/\langle x \rangle^{\alpha+\frac{3}{2}}$.

PROOF: We first notice that the two exponential terms in (5.3) can be written as

$$(5.13) \quad e^{-\frac{(x-ry)^2}{1-r^2}} e^{\frac{x^2-y^2}{2}} = e^{-\frac{1+r^2}{1-r^2} \frac{x^2+y^2}{2}} e^{\frac{2rxy}{1-r^2}} \lesssim e^{-\frac{x^2+y^2}{2}},$$

since $\frac{1+r^2}{1-r^2} \geq 1$ and in the region of integration of $B_t(x, y)$ the exponent $z = \frac{2rxy}{1-r^2} \lesssim 1$. We now separate cases.

(i) *Case $xy \geq 1$:* then $r_0(xy) = \frac{1}{2xy} \leq \frac{1}{2}$ and

$$(5.14) \quad B_t(x, y) \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{(\ln \frac{1}{r})^{\nu+1}} \frac{dr}{r}.$$

The last integral is approximately given by

$$\int_0^{\frac{1}{2xy}} \frac{r^{\alpha+\mu+1}}{(\ln \frac{1}{r})^{\nu+1}} \frac{dr}{r} \approx \left(\frac{1}{xy}\right)^{\alpha+\mu+1} \frac{1}{[\log(2xy)]^{1+\nu}}$$

(with the usual convention when $\alpha + \mu + 1 = 0$ of reducing the log by one power). This is a good estimate if we assume that $x \geq 1/2$, since we may use

$$\log(2xy) \gtrsim \log(y \vee 2) \approx \log(y + e),$$

and overall obtain

$$B_t(x, y) \lesssim t^{2\nu} x^{-(\mu+\frac{1}{2})} e^{-\frac{x^2}{2}} \frac{y^{\alpha+\frac{1}{2}} e^{-y^2/2}}{y^{\alpha+\mu+1} [\log(y + e)]^{1+\nu}} \lesssim t^{2\nu} \Phi(y).$$

When $x \leq 1/2$, we need a refinement to obtain the $c(x)$ in the statement of the lemma. We split the integral defining $B_t(x, y)$ as

$$(5.15) \quad B_t(x, y) = \int_0^{\frac{2x}{y}} \cdots + \int_{\frac{2x}{y}}^{r_0(xy)} \cdots = I + II,$$

noticing that the partition point $\frac{2x}{y} \leq r_0(xy) = \frac{1}{2xy}$. Since $x \leq \frac{1}{2}$ and $xy \geq 1$ we also have $y \geq 2$. Now, the first integral can be bound as above by

$$\begin{aligned} I &\lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \left(\frac{x}{y}\right)^{\alpha+\mu+1} \frac{1}{[\log(\frac{y}{2x})]^{1+\nu}} \\ &\lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} x^{\alpha+\mu+1} e^{-\frac{x^2}{2}} \frac{e^{-y^2/2}}{y^{\mu+\frac{1}{2}} [\log y]^{1+\nu}} \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y). \end{aligned}$$

since in this range $y \geq 2$. This implies the stated estimate because $\langle x \rangle^{\alpha+\frac{1}{2}} \leq 1/\langle x \rangle^{\alpha+\frac{3}{2}} = c(x)$. To handle II we need a different bound for the exponentials in (5.13), noticing that

$$(5.16) \quad r > \frac{2x}{y} \quad \Rightarrow \quad |x - ry| = ry - x \geq \frac{ry}{2} \quad \Rightarrow \quad e^{-\frac{(x-ry)^2}{1-r^2}} \leq e^{-\frac{r^2 y^2}{4}}.$$

Thus

$$(5.17) \quad II \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{\frac{x^2-y^2}{2}} \int_{\frac{2x}{y}}^{\frac{1}{2}} \frac{r^{\alpha+\mu+1} e^{-\frac{(ry)^2}{4}}}{\left(\ln \frac{1}{r}\right)^{\nu+1}} \frac{dr}{r}.$$

Changing variables $ry = u$, the latter integral can be estimated by

$$y^{-(\alpha+\mu+1)} \int_0^{\frac{y}{2}} \frac{u^{\alpha+\mu+1} e^{-\frac{u^2}{4}}}{\left(\ln \frac{y}{u}\right)^{\nu+1}} \frac{du}{u} \approx \frac{1}{y^{\alpha+\mu+1} [\log y]^{1+\nu}}$$

since the major contribution happens when $u \approx 1$ (with the usual convention of reducing a log power if $\alpha + \mu + 1 = 0$). Inserting this into (5.17) (and using $y \geq 2$ and $x \leq 1/2$) we obtain once again

$$II \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y).$$

This concludes the proof of the case $xy \geq 1$.

(ii) *Case* $xy \leq 1$: this time $r_0(xy) = 1 - \frac{xy}{2} \geq \frac{1}{2}$, so we may split

$$B_t(x, y) = \int_0^{\frac{1}{2}} \cdots + \int_{\frac{1}{2}}^{r_0(xy)} \cdots = B_1 + B_2$$

The first term can be handled essentially as in the previous case. Namely, if $y \leq 2$ we use a similar bound to (5.14)

$$B_1 \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} \int_0^{\frac{1}{2}} \frac{r^{\alpha+\mu+1}}{\left(\ln \frac{1}{r}\right)^{\nu+1}} \frac{dr}{r} \approx t^{2\nu} x^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}} \langle y \rangle^{\alpha+\frac{1}{2}} \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y).$$

If $y \geq 2$, then $x \leq \frac{1}{2}$ and $\frac{2x}{y} \leq \frac{1}{2}$, so we may split

$$B_1 \leq \int_0^{\frac{2x}{y}} \cdots + \int_{\frac{2x}{y}}^{\frac{1}{2}} \cdots$$

and exactly the same computations we used in (5.15), give us the bound

$$B_1 \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y).$$

Thus we are left with the integral corresponding to B_2 , that is the range $\frac{1}{2} < r < 1 - \frac{xy}{2}$. First of all, observe that

$$\ln \frac{1}{r} \approx 1 - r, \quad r \in [1/2, 1] \quad \Rightarrow \quad e^{-\frac{t^2}{2 \ln \frac{1}{r}}} \leq e^{-\frac{ct^2}{1-r}},$$

for a suitable $c > 0$. Next, we need once again more precise bounds for the exponentials in (5.13). We claim that, if $r \in [1/2, 1]$ then

$$e^{-\frac{1+r^2}{1-r^2} \frac{x^2+y^2}{2}} \leq e^{-\gamma \frac{x^2+y^2}{1-r}} e^{-(1+\gamma) \frac{x^2+y^2}{2}}$$

for a small constant $\gamma > 0$. This is easily obtained using the fact that $\frac{1+r^2}{1-r^2} \geq \frac{5}{3}$ in this interval. With these exponential bounds we can control the integral B_2 as follows

$$(5.18) \quad \begin{aligned} B_2 &\lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-(1+\gamma)\frac{x^2+y^2}{2}} \int_{1/2}^{1-\frac{xy}{2}} \frac{e^{-\frac{ct^2+\gamma(x^2+y^2)}{1-r}}}{(1-r)^{\alpha+\nu+1}} \frac{dr}{1-r} \\ &\lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{[t^2+x^2+y^2]^{\alpha+\nu+1}} e^{-(1+\gamma)\frac{x^2+y^2}{2}} \int_0^\infty e^{-u} u^{\alpha+\nu+1} \frac{du}{u}, \end{aligned}$$

after changing variables $u = [ct^2 + \gamma(x^2 + y^2)]/(1-r)$. The last integral is a finite constant (because $\alpha + \nu + 1 > 0$), so we observe three possible cases:

(1) if $y \geq 1$, we can disregard the denominator and obtain

$$B_2 \lesssim t^{2\nu} (xy)^{\alpha+\frac{1}{2}} e^{-(1+\gamma)\frac{y^2}{2}} \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \Phi(y),$$

since the exponential decay in y is actually better than $\Phi(y)$ (and also $x \leq 1$).

(2) if $y \leq 1$ and $\max\{x, t\} \geq 1$, we can also disregard the denominator and obtain

$$B_2 \lesssim t^{2\nu} \langle y \rangle^{\alpha+\frac{1}{2}} x^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}} \lesssim t^{2\nu} \langle x \rangle^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}}.$$

(3) if all $y, t, x \leq 1$, we bound the denominator in the two obvious ways to obtain

$$(5.19) \quad B_2 \lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{t^{2\nu} x^{2(\alpha+1)}} = \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}},$$

which is precisely the upper bound stated in (5.12). Observe that when $x \rightarrow 0$ this piece gives the largest contribution to $B_t(x, y)$. \square

Remark 5.6. It is also possible to obtain a bound

$$(5.20) \quad B_t(x, y) \lesssim c'(x) t^{2\nu} \Phi(y),$$

with perhaps a worse function $c'(x)$, but without the loss produced by $\max\{1, t^{2\nu}\}$. This loss appeared when $t, x, y \leq 1$ in (5.19) above. Looking at (5.18) we may replace that bound by

$$B_2 \lesssim \frac{t^{2\nu} (xy)^{\alpha+\frac{1}{2}}}{x^{2(\alpha+\nu+1)}},$$

which implies (5.20) with $c'(x) = 1/lx^{\alpha+\frac{3}{2}+2\nu}$. This estimate will also be useful later.

5.3. Upper estimates for $A_t(x, y)$: integrals over $r \leq 1/2$. Recall that

$$(5.21) \quad A_t(x, y) \approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{r_0(xy)}^1 \frac{r^{\mu+\frac{1}{2}}}{\sqrt{1-r} (\ln \frac{1}{r})^{1+\nu}} e^{-\frac{t^2}{2 \ln \frac{1}{r}}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r}.$$

When $xy \geq 1$ we have $r_0(xy) = \frac{1}{2xy} \leq \frac{1}{2}$, so we can write

$$A_t(x, y) = \int_{r_0(xy)}^{\frac{1}{2}} \cdots + \int_{\frac{1}{2}}^1 \cdots = A1 + A2.$$

In this section we shall prove the following estimate for $A1$.

Lemma 5.7. *If $xy \geq 1$, then*

$$(5.22) \quad A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{(\ln \frac{1}{r})^{1+\nu}} e^{-\frac{(x-ry)^2}{1-r^2}} \frac{dr}{r} \lesssim c(x) t^{2\nu} \Phi(y),$$

where $c(x) = [\log(e+x)]^{\nu+1} (1+x)^{|\mu+\frac{1}{2}|} \exp(x^2/2)$.

PROOF: We shall distinguish cases

(i) *Case $y \leq 4x$.* In this region we essentially disregard the exponential term $e^{-\frac{(x-ry)^2}{1-r^2}}$ inside the integral, and directly estimate

$$(5.23) \quad A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r}.$$

Notice however that when $y \leq x$ the exponential produces an additional gain, due to

$$(5.24) \quad ry \leq \frac{x}{2} \Rightarrow |x-ry| \geq \frac{x}{2} \Rightarrow e^{-\frac{(x-ry)^2}{1-r^2}} \leq e^{-\frac{x^2}{4}}.$$

This will play a role later in evaluating the constant $c(x)$. We now evaluate the integral in (5.23), depending on the sign of $\mu + \frac{1}{2}$.

(1) If $\mu + \frac{1}{2} \geq 0$ the integral is bounded by a constant, and hence

$$A1 \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}}.$$

We shall enlarge this value to match (5.22) as follows. Since $xy \geq 1$, in this range we have $x \geq 1/2$. So if $1 \leq y \leq 4x$ we may use

$$1 \lesssim \frac{(1+x)^{\mu+\frac{1}{2}} [\log(e+x)]^{\nu+1}}{(1+y)^{\mu+\frac{1}{2}} [\log(e+y)]^{\nu+1}}.$$

If $\frac{1}{x} \leq y \leq 1$, we use instead

$$1 \lesssim \max\{x^{\alpha+\frac{1}{2}}, 1\} \langle y \rangle^{\alpha+\frac{1}{2}},$$

which in this region can be combined with the extra exponential in factor in (5.24).

In both cases we obtain $A1 \lesssim t^{2\nu} c(x) \Phi(y)$, as wished.

(2) If $\mu + \frac{1}{2} < 0$ the integral diverges near 0, but we still obtain

$$\int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r} \lesssim \frac{1}{(xy)^{\mu+\frac{1}{2}} (\log 2xy)^{\nu+1}}.$$

Thus, using the inequality $\log(2xy) \gtrsim \max\{\log y, \log 2\}$ we arrive at

$$\begin{aligned} A1 &\lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \frac{(1+x)^{|\mu+\frac{1}{2}|}}{(1+y)^{\mu+\frac{1}{2}} [\log(e+y)]^{\nu+1}} \leq t^{2\nu} c(x) \Phi(y), \quad \text{if } 1 \leq y \leq 4x \\ &\lesssim t^{2\nu} \frac{e^{\frac{x^2}{4}} y^{\alpha+\frac{1}{2}}}{x^{\mu+\frac{1}{2}} y^{\mu+\alpha+1}} \leq t^{2\nu} e^{\frac{x^2}{4}} x^{\alpha+\frac{1}{2}} \langle y \rangle^{\alpha+\frac{1}{2}} \leq t^{2\nu} c(x) \Phi(y), \quad \text{if } \frac{1}{x} \leq y \leq 1, \end{aligned}$$

using in the last case the additional exponential gain in (5.24).

(ii) *Case* $y \geq 4x$. This is the same as $\frac{2x}{y} \leq \frac{1}{2}$, and remember from (5.16) that when $r \in [\frac{2x}{y}, \frac{1}{2}]$ a better bound for the exponential is available, namely

$$(5.25) \quad e^{-\frac{(x-ry)^2}{1-r^2}} \leq e^{-\frac{r^2 y^2}{4}}.$$

Thus we may consider two subcases, depending on whether $\frac{2x}{y}$ is above or below $r_0(xy)$.

• *Subcase* $\frac{2x}{y} \leq r_0(xy) = \frac{1}{2xy} \leq \frac{1}{2}$. Using (5.25) we obtain

$$\begin{aligned} A1 &\lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{1}{2}} \frac{r^{\mu+\frac{1}{2}} e^{-\frac{(ry)^2}{4}}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r} \\ (ry = u) &= \frac{t^{2\nu} e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{\frac{1}{2x}}^{\infty} \frac{u^{\mu+\frac{1}{2}} e^{-\frac{u^2}{4}}}{(\ln \frac{y}{u})^{1+\nu}} \frac{du}{u} \end{aligned}$$

Observe that $x \leq \frac{1}{2}$ (and $y \geq 2$), so the latter integral reaches its major contribution at $u = \frac{1}{2x}$

$$\int_{\frac{1}{2x}}^{\infty} \frac{u^{\mu+\frac{1}{2}} e^{-\frac{u^2}{4}}}{(\ln \frac{y}{u})^{1+\nu}} \frac{du}{u} \lesssim \frac{(1/x)^{\mu-\frac{1}{2}} e^{-c/x^2}}{(\log 2xy)^{1+\nu}} \lesssim \frac{1}{(\log y)^{1+\nu}},$$

using in the last step the elementary bound of logarithms

$$\log(2xy) \gtrsim \frac{\log(y+e)}{\log(\frac{1}{x}+e)}, \quad \text{if } y \geq \max\{1, 1/x\}$$

(see e.g. [4, Lemma 5.1]). Thus we conclude that

$$A1 \lesssim t^{2\nu} e^{\frac{x^2}{2}} \Phi(y).$$

• *Subcase* $r_0(xy) < \frac{2x}{y} \leq \frac{1}{2}$. Here we split

$$A1 = \int_{\frac{2x}{y}}^{\frac{1}{2}} \dots + \int_{r_0(xy)}^{\frac{2x}{y}} \dots = I + II.$$

The first term is similar to the previous subcase, except that now $x > \frac{1}{2}$ (and $y \geq 4x \geq 2$)

$$I \lesssim \frac{t^{2\nu} e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{\frac{x}{2}}^{\infty} \frac{u^{\mu+\frac{1}{2}} e^{-\frac{u^2}{4}}}{(\ln \frac{y}{u})^{1+\nu}} \frac{du}{u}$$

and the last integral is bounded by a constant times

$$\frac{x^{\mu-\frac{1}{2}} e^{-cx^2}}{(\ln \frac{2y}{x})^{1+\nu}} \lesssim x^{\mu-\frac{1}{2}} e^{-cx^2} \frac{[\log(e+x)]^{1+\nu}}{[\log(e+y)]^{1+\nu}} \lesssim \frac{1}{[\log(e+y)]^{1+\nu}}.$$

Finally, we consider II . Here there is no exponential gain, and similarly to (5.23) we have

$$II \lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\frac{1}{2xy}}^{\frac{2x}{y}} \frac{r^{\mu+\frac{1}{2}}}{(\ln \frac{1}{r})^{1+\nu}} \frac{dr}{r} = \frac{t^{2\nu} e^{\frac{x^2-y^2}{2}}}{y^{\mu+\frac{1}{2}}} \int_{\frac{1}{2x}}^{2x} \frac{u^{\mu+\frac{1}{2}}}{(\ln \frac{y}{u})^{1+\nu}} \frac{du}{u}.$$

Now, the last integral can easily be analyzed (depending on the sign of $\mu + \frac{1}{2}$) to obtain

$$\int_{\frac{1}{2x}}^{2x} \frac{u^{\mu+\frac{1}{2}}}{(\ln \frac{y}{u})^{1+\nu}} \frac{du}{u} \lesssim \frac{x^{|\mu+\frac{1}{2}|} [\log(x+e)]^{1+\nu}}{[\log(y+e)]^{1+\nu}}.$$

Thus, overall we conclude that in this subcase

$$A1 \lesssim I + II \lesssim t^{2\nu} (1+x)^{|\mu+\frac{1}{2}|} [\log(x+e)]^{1+\nu} e^{\frac{x^2}{2}} \Phi(y).$$

□

5.4. Upper estimates for $A_t(x, y)$ when $y \leq x/2$ or $y \geq Mx$. In view of the previous subsection, it only remains to estimate

$$(5.26) \quad \begin{aligned} A2 &\approx t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\max\{r_0(xy), \frac{1}{2}\}}^1 \frac{e^{-\frac{t^2}{2 \ln \frac{1}{r}}}}{\sqrt{1-r} (\ln \frac{1}{r})^{1+\nu}} e^{-\frac{(x-ry)^2}{1-r^2}} dr \\ &\lesssim t^{2\nu} e^{\frac{x^2-y^2}{2}} \int_{\max\{1-\frac{xy}{2}, \frac{1}{2}\}}^1 \frac{e^{-\frac{ct^2}{1-r}}}{(1-r)^{\nu+\frac{3}{2}}} e^{-\frac{(x-ry)^2}{1-r^2}} dr \end{aligned}$$

noticing that $\ln \frac{1}{r} \approx 1-r$ when $r \in [\frac{1}{2}, 1]$. In this region, however, it is more convenient to use the write-up for the heat kernel in (5.4), in terms of the parameter s . This gives a more reasonable expression for the exponentials, namely

$$e^{\frac{x^2-y^2}{2}} e^{-\frac{(x-ry)^2}{1-r^2}} = e^{-\frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]}.$$

Since the parameters r and s are related by $s = \frac{1-r}{1+r}$ (or $r = \frac{1-s}{1+s}$), either from (5.4) or directly from (5.26), we obtain that

$$(5.27) \quad A2 \lesssim t^{2\nu} \int_0^{\min\{\frac{1}{3}, \frac{xy}{3}\}} \frac{e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]}}{s^{\nu+\frac{3}{2}}} ds,$$

after perhaps slightly enlarging the range of integration. Our first result shows that when y is far from x this can also be controlled by the function $\Phi(y)$.

Lemma 5.8. *There exists $M > 1$ such that, if $y \leq \frac{x}{2}$ or $y \geq Mx$, then*

$$A2 \lesssim t^{2\nu} \int_0^{\min\{\frac{1}{3}, \frac{xy}{3}\}} \frac{e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]}}{s^{\nu+\frac{3}{2}}} ds \lesssim c(x) \max\{t^{2\nu}, 1\} \Phi(y),$$

with $c(x) = 1/\langle x \rangle^{\alpha+\frac{3}{2}}$.

PROOF: We claim that, in the assumed range of x and y , there is some $\gamma > 0$ such that

$$(5.28) \quad A2 \lesssim t^{2\nu} e^{-(\frac{1}{2}+\gamma)y^2} \int_0^{\min\{\frac{1}{3}, \frac{xy}{3}\}} e^{-\gamma \frac{t^2+(x-y)^2}{s}} s^{-(\nu+\frac{3}{2})} ds$$

This is just a bound of the exponentials. Indeed, if we distinguish the two cases

(1) *case $y \geq Mx$:* this implies $|y-x| \geq (1-\frac{1}{M})y$, so for any $\eta < 1$ we have

$$e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]} \leq e^{-\frac{ct^2+\frac{\eta}{4}(x-y)^2}{s}} e^{-\frac{1-\eta}{4}(\frac{M-1}{M})^2(\frac{1}{s}+s)y^2}$$

which implies the required assertion using that $\frac{1}{s} + s \geq \frac{10}{3}$, when $s \in (0, \frac{1}{3})$, and choosing M sufficiently large and η sufficiently small.

(2) *case* $y \leq x/2$: this time $|x - y| \geq \frac{x}{2} \geq y$, so we have

$$e^{-\frac{ct^2}{s}} e^{-\frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]} \leq e^{-\frac{ct^2 + \frac{\eta}{4}(x-y)^2}{s}} e^{-\frac{1-\eta}{4}(\frac{1}{s}+s)y^2}$$

which again implies the assertion using $\frac{1}{s} + s \geq \frac{10}{3}$ and choosing η sufficiently small.

Thus (5.28) is proven, and we may change variables $[t^2 + (x - y)^2]/s = u$ to obtain

$$(5.29) \quad A2 \lesssim \frac{t^{2\nu} e^{-(\frac{1}{2}+\gamma)y^2}}{[t^2 + (x - y)^2]^{\nu+\frac{1}{2}}} \int_{3\gamma[t^2+(x-y)^2] \max\{1, \frac{1}{xy}\}}^{\infty} u^{\nu+\frac{1}{2}} e^{-u} \frac{du}{u}$$

$$(5.30) \quad \lesssim \frac{t^{2\nu} e^{-(\frac{1}{2}+\gamma)y^2}}{(t+x+y)^{2\nu+1}} \int_{\gamma'[x^2+y^2] \max\{1, \frac{1}{xy}\}}^{\infty} u^{\nu+\frac{1}{2}} e^{-u} \frac{du}{u},$$

since in the selected range of x, y we have $|x - y| \gtrsim x + y$. To finish the proof we must distinguish some cases.

Case $y \geq 1$: then bounding the denominator and the integral in (5.30) by a constant we immediately see that

$$A2 \lesssim t^{2\nu} e^{-(\frac{1}{2}+\gamma)y^2} \lesssim t^{2\nu} \Phi(y)$$

since the exponential has a better decay.

Case $y \leq 1$ and $y \geq Mx$: we again bound the integral by a constant and estimate the fraction in (5.30) as follows

$$(5.31) \quad A2 \lesssim \frac{t^{2\nu}}{t^{2\nu}y} = \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}} \leq c_M \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}.$$

Case $y \leq 1$ and $y \leq \frac{x}{2}$: this is a relevant case, since the integral in (5.30) plays actually a role. To compute the integral we must distinguish the two subcases

(1) If $xy \leq 1$, then since also $\frac{x}{y} \geq 2$,

$$(5.32) \quad \begin{aligned} A2 &\lesssim \frac{t^{2\nu}}{t^{2\nu}x} \int_{\gamma' \frac{x}{y}}^{\infty} u^{\nu-\frac{1}{2}} e^{-u} du \approx \frac{1}{x} \left(\frac{x}{y}\right)^{\nu-\frac{1}{2}} e^{-c \frac{x}{y}} \\ &\lesssim \frac{1}{x} \left(\frac{y}{x}\right)^{\alpha+\frac{1}{2}} \leq \frac{\langle y \rangle^{\alpha+\frac{1}{2}}}{\langle x \rangle^{\alpha+\frac{3}{2}}}. \end{aligned}$$

(2) If $xy \geq 1$, then we have $x \geq \frac{1}{y} \geq 1$ and

$$A2 \lesssim \frac{t^{2\nu}}{x^{1+2\nu}} \int_{\gamma' x^2}^{\infty} u^{\nu-\frac{1}{2}} e^{-u} du \lesssim t^{2\nu} x^{-2} e^{-cx^2}.$$

Now, since $\frac{1}{x} \leq y \leq 1$ we can insert the estimate

$$1 \lesssim \langle y \rangle^{\alpha+\frac{1}{2}} \max\{x^{\alpha+\frac{1}{2}}, 1\},$$

to obtain $A2 \lesssim t^{2\nu} \langle y \rangle^{\alpha+\frac{1}{2}}$. □

Remark 5.9. As mentioned earlier in Remark 5.6, here it is also possible to obtain a bound

$$(5.33) \quad A2 \lesssim c'(x) t^{2\nu} \Phi(y),$$

with $c'(x) = 1/lx^{\alpha+\frac{3}{2}+2\nu}$. The loss produced by $\max\{1, t^{2\nu}\}$ can be corrected in (5.31) and (5.32) by replacing the factor $t^{2\nu}$ in the denominator by $x^{2\nu}$, as one readily notices from (5.30). As mentioned before, this estimate will play a role later.

5.5. Upper estimates for $A_t(x, y)$ in the local part $\frac{x}{2} < y < Mx$. As in the previous subsection, our starting point is the formula (5.27), which we must estimate in the local region $\frac{x}{2} < y < Mx$. A sufficient bound for us is stated in the next lemma.

Lemma 5.10. *If $\frac{x}{2} < y < Mx$, then*

$$(5.34) \quad t^{2\nu} \int_0^{\min\{\frac{1}{3}, \frac{xy}{3}\}} \frac{e^{-\frac{ct^2}{s} - \frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]}}{s^{\nu+\frac{3}{2}}} ds \lesssim C(x) \frac{t^{2\nu} e^{-\frac{y^2}{2}}}{(t + |x - y|)^{1+2\nu}},$$

where $C(x) = (1+x)^{2\nu} e^{\frac{x^2}{2}}$.

PROOF: We shall crudely enlarge the integral in (5.34) to the range $\int_0^{1/3}$. This last integral was already estimated in [5] and [4], by a similar procedure to the one used in the last subsection. More precisely, from the estimates in [4, Lemma 3.2], formula (3.16), it follows that

$$t^{2\nu} \int_0^{\frac{1}{2}} \frac{e^{-\frac{ct^2}{s} - \frac{1}{4}[\frac{(x-y)^2}{s} + s(x+y)^2]}}{s^{\nu+\frac{3}{2}}} ds \lesssim \frac{t^{2\nu} (1+x)^{2\nu} e^{\frac{x^2-y^2}{2}}}{(t + |x - y|)^{1+2\nu}},$$

which agrees with (5.34). \square

5.6. Proof of Propositions 5.1 and 5.2. Proposition 5.2 follows by putting together Lemmas 5.5, 5.7, 5.8 and 5.10. Concerning Proposition 5.1, the lower bound was shown in Lemma 5.4, while the upper bound also follows from Lemmas 5.5, 5.7 and 5.8, at least when $y < \frac{x}{2}$ or $y > Mx$. This actually implies the asserted result for all x and y , since when y belongs to the compact set $[\frac{x}{2}, Mx]$, the continuous function $y \rightarrow P_t(x, y)/\Phi(y)$ is bounded above by a constant $c_2(t, x)$. \square

5.7. Proof of Corollary 5.3. By Proposition 5.2, observe that

$$(5.35) \quad P_t f(x) \lesssim \frac{C_1(x)}{t} \int_{\mathbb{R}_+} \frac{g(y) dy}{(1 + \frac{|x-y|}{t})^{1+2\nu}} + C_2(x)(1 \vee t_0)^{2\nu} \|f\|_{L^1(\Phi)},$$

where $g(y) = f(y)e^{-\frac{y^2}{2}} \chi_{\{\frac{x}{2} < y < Mx\}}$. The first term is then controlled by a maximal function by a standard slicing argument. \square

Remark 5.11. We wrote in (1.9) a different version of (5.10) with $\mathcal{M}^{\text{loc}}(f\Phi)$ in place of $\mathcal{M}^{\text{loc}}(fe^{-\frac{y^2}{2}})$. Since $x \approx y$,

$$\mathcal{M}^{\text{loc}}(f\Phi)(x) \approx \frac{\langle x \rangle^{\alpha+\frac{1}{2}}}{[\log(e+x)]^{1+\nu} (1+x)^{\mu+\frac{1}{2}}} \mathcal{M}^{\text{loc}}(fe^{-\frac{y^2}{2}})(x),$$

so they are actually equivalent modulo x -constants. The write-up in (1.9) has the advantage of remaining valid for other Laguerre systems; see §7 below.

6. PROOFS

As indicated in §1 we present the proof of Theorems 1.1 and 1.2 for the differential operator L in (5.1) and the function Φ in (5.6). We postpone to §7 the proof of the results for the other systems mentioned in Table 1.

6.1. Proof of Theorem 1.1. First of all, it is an immediate consequence of Proposition 5.1 that $P_t|f|(x) < \infty$ for some (or all) $t, x > 0$ if and only if $f \in L^1(\Phi)$. This justifies that $f \in L^1(\Phi)$ is the right setting for this problem. Notice also that taking derivatives of the kernel $P_t(x, y)$ in (5.5) with respect to t does not worsen its decay in y , so $P_t f(x)$ automatically becomes infinitely differentiable in the t -variable when $f \in L^1(\Phi)$. We can also take as many derivatives as wished with respect to x , since the kernel satisfies the pde[‡]

$$\left[\partial_{xx} - x^2 - \frac{\alpha^2 - \frac{1}{4}}{x^2} - 2\mu + \partial_{tt} + \frac{1-2\nu}{t} \partial_t \right] P_t(x, y) = 0,$$

so x -derivatives are transformed into t -derivatives and do not worsen the decay of $P_t(x, y)$ in the y -variable. We have thus completed the proof of paragraph (i) and the last statement in Theorem 1.1. We shall now prove a stronger result than (ii), namely that for $f \in L^1(\Phi)$

$$(6.1) \quad \lim_{t \rightarrow 0^+} P_t f(x) = f(x), \quad \forall x \in \mathcal{L}_f$$

where \mathcal{L}_f denotes the set of Lebesgue points of f . When $f(x) = 0$ this is easily obtained from the kernel estimates in Proposition 5.2. Indeed,

$$P_t f(x) \lesssim C_1(x) \int_{\frac{x}{2}}^{Mx} \frac{t^{2\nu} |f(y)| dy}{(t + |x - y|)^{1+2\nu}} + C'_2(x) t^{2\nu} \int_{\mathbb{R}_+} |f| \Phi,$$

where in the second term we have replaced $(t \vee 1)^{2\nu}$ by $t^{2\nu}$ in view of Remarks 5.20 and 5.33. Thus, this second term vanishes as $t \rightarrow 0$ (actually for all $x \in \mathbb{R}_+$). Concerning the first term, it is given by convolution of $|f(y)| \chi_{\{\frac{x}{2} < y < Mx\}} \in L_c^1(\mathbb{R}_+)$ with a radially decreasing approximate identity, so from well-known results (see e.g. [13, p. 112]), it must vanish as $t \rightarrow 0$ at every Lebesgue point x of f with $f(x) = 0$.

It remains to prove (6.1) when $f(x)$ is not necessarily 0. To show this, we first notice that the first eigenfunction $\varphi = \varphi_0^\alpha$ (with eigenvalue $\lambda = 2(\mu + \alpha + 1)$) satisfies

$$P_t \varphi = F_t(\lambda) \varphi, \quad \text{with} \quad \lim_{t \rightarrow 0} F_t(\lambda) = 1.$$

Indeed, setting $u = t^2/(4v)$ in (1.3), gives

$$F_t(\lambda) = \frac{(t/2)^{2\nu}}{\Gamma(\nu)} \int_0^\infty e^{-\frac{t^2}{4u} - \lambda u} \frac{du}{u^{1+\nu}} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-v - \frac{t^2 \lambda}{4v}} v^{\nu-1} dv \longrightarrow 1, \quad \text{as } t \rightarrow 0.$$

Therefore, we can write

$$(6.2) \quad P_t f(x) - f(x) = P_t f(x) - F_t(\lambda) f(x) + f(x) [F_t(\lambda) - 1],$$

[‡]For a justification that the subordinated integral in (1.3) satisfies the pde (1.2), see e.g. [4, §2].

with the last term vanishing as $t \rightarrow 0$. Since $\varphi > 0$, the first term can be rewritten as

$$P_t f(x) - \frac{f(x)}{\varphi(x)} P_t \varphi(x) = P_t \left(f - \frac{f(x)}{\varphi(x)} \varphi \right) (x).$$

Setting $g = f - \frac{f(x)}{\varphi(x)} \varphi$, it is easily seen that $g \in L^1(\Phi)$, $g(x) = 0$ and x is a Lebesgue point of g . This last assertion follows from

$$\int_{I_r(x)} |g(y)| dy \leq \int_{I_r(x)} |f(y) - f(x)| dy + \frac{|f(x)|}{\varphi(x)} \int_{I_r(x)} |\varphi(y) - \varphi(x)| dy,$$

which vanishes as $r \rightarrow 0$. Thus we can apply our earlier case to g and conclude that $\lim_{t \rightarrow 0} P_t g(x) = 0$. So the left hand side of (6.2) goes to 0 as $t \rightarrow 0$, establishing (6.1) and completing the proof of Theorem 1.1. \square

Remark 6.1. A close look at the last part of the proof shows that, when $f \in C([a, b])$ with $[a, b] \subseteq \mathbb{R}_+$, then the convergence of $P_t f(x) \rightarrow f(x)$ is uniform in $x \in [a, b]$.

6.2. Proof of Theorem 1.2. We have to show that $P_{t_0}^*$ maps $L^p(w) \rightarrow L^p(v)$, for some weight $v(x) > 0$, under the assumption that

$$\|w\|_{D_p(\Phi)} := \left[\int_{\mathbb{R}_+} w^{-\frac{1}{p-1}}(x) \Phi(x)^{p'} dx \right]^{1/p'} < \infty,$$

with Φ defined as in (5.6). We shall use the bound for $P_{t_0}^*$ in (5.10), namely

$$\begin{aligned} P_{t_0}^* f(x) &\lesssim C_1(x) \mathcal{M}_M^{\text{loc}}(f e^{-\frac{y^2}{2}})(x) + C_2(x) \int_{\mathbb{R}_+} |f(y)| \Phi(y) dy \\ (6.3) \quad &= I(x) + II(x), \end{aligned}$$

for a suitable $M > 1$, and $C_1(x), C_2(x)$ given explicitly in Proposition 5.2. We first treat the last term, which by Hölder's inequality is bounded by

$$II(x) \leq C_2(x) \|f\|_{L^p(w)} \|w\|_{D_p(\Phi)}.$$

Thus, it suffices to choose a weight v such that $C_2(x) = [\log(e+x)]^{1+\nu} (1+x)^{|\mu+\frac{1}{2}|} e^{\frac{x^2}{2}} / \langle x \rangle^{\alpha+\frac{3}{2}}$ belongs to $L^p(v)$ to conclude that

$$(6.4) \quad \|II\|_{L^p(v)} \leq \|C_2\|_{L^p(v)} \|w\|_{D_p(\Phi)} \|f\|_{L^p(w)}.$$

For instance we may take any $v(x) \leq v_2(x)$ with

$$(6.5) \quad v_2(x) := \frac{l x^{(\alpha+\frac{3}{2})p-1}}{[\log(e/lx)]^2} \frac{e^{-\frac{p}{2}x^2}}{(1+x)^N}$$

for any $N > 1 + p|\mu + \frac{1}{2}|$. We remark that $v_2 \in D_q(\Phi)$ for all $q > p$. Indeed, the local condition was already established in (4.3). For the global condition notice that

$$\int_1^\infty v_2(x)^{-\frac{q'}{q}} \Phi(x)^{q'} dx \lesssim \int_1^\infty e^{-(1-\frac{p}{q})\frac{q'}{2}x^2} (1+x)^{\frac{Nq'}{q}} dx < \infty.$$

We now consider the term $I(x)$ in (6.3). We define a new weight $W(x) = w(x)e^{\frac{p}{2}x^2}$, and observe that

$$(6.6) \quad w \in D_p(\Phi) \implies W \in D_p^0(\alpha + \frac{1}{2}) \cap D_p^{\text{exp}}(a), \quad \forall a > 0,$$

for the weight classes defined just before Proposition 3.2. Indeed, the local estimate follows from $\Phi(x) \approx \langle x \rangle^{\alpha+\frac{1}{2}}$ when $x \in (0, 1)$, and the global estimate is a consequence of

$$(6.7) \quad \|W\|_{D_p^{\text{exp}}(a)} = \int_1^\infty w^{-\frac{p'}{p}}(x) e^{-\frac{p'}{2}x^2} e^{-ap'x^2} dx \leq C_a^{p'} \int_1^\infty w^{-\frac{p'}{p}}(x) \Phi(x)^{p'} dx,$$

with $C_a = \max_{x \geq 1} |\log(e+x)|^{1+\nu} |1+x|^{\mu+\frac{1}{2}} e^{-ax^2} < \infty$.

We shall now set

$$(6.8) \quad v_{1,\varepsilon}(x) = \frac{e^{-\frac{px^2}{2}}}{(1+x)^{2p\nu}} \mathcal{V}(x) \rho_\varepsilon(\mathcal{V}(x)), \quad \text{where } \mathcal{V}(x) = \left[\mathcal{M}_M^{\text{loc}}(w^{-\frac{p'}{p}} e^{-\frac{p'}{2}x^2})(x) \right]^{-\frac{p}{p'}}$$

(or $v_{1,\varepsilon}(x) = (1+x)^{-2p\nu} e^{-\frac{px^2}{2}} V_\varepsilon(x)$ in the notation of (3.1)). Given $f \in L^p(w)$, we denote $\tilde{f}(y) = f(y) e^{-\frac{y^2}{2}}$ which is a function in $L^p(W)$. Then, using the two-weight inequality for $\mathcal{M}_M^{\text{loc}}$ in Theorem 3.1, and the expression for $C_1(x) = (1+x)^{2\nu} e^{\frac{x^2}{2}}$, we see that, for any $v \leq v_{1,\varepsilon}$, the term $I(x)$ in (6.3) is controlled by

$$(6.9) \quad \begin{aligned} \|I(x)\|_{L^p(v)}^p &\leq \int_{\mathbb{R}_+} \frac{C_1(x)^p e^{-\frac{p}{2}x^2}}{(1+|x|)^{p2\nu}} |\mathcal{M}_M^{\text{loc}} \tilde{f}(x)|^p V_\varepsilon(x) dx \\ &\lesssim \|\tilde{f}\|_{L^p(W)}^p = \|f\|_{L^p(w)}^p. \end{aligned}$$

So, combining (6.3), (6.4) and (6.9) we have shown that $\|P_{t_0}^* f\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, provided

$$(6.10) \quad v(x) = \min\{v_{1,\varepsilon}(x), v_2(x)\},$$

with $v_{1,\varepsilon}(x)$ and $v_2(x)$ defined in (6.8) and (6.5).

We only have to verify that, if $q > p$ then we can choose ε sufficiently small so that $v_{1,\varepsilon} \in D_q(\Phi)$ (which implies $v \in D_q(\Phi)$). This actually follows from (6.6) and Proposition 3.2. Indeed, on the one hand, since $W \in D_p^0(\alpha + \frac{1}{2})$

$$(6.11) \quad \int_0^1 v_{1,\varepsilon}(x)^{-\frac{q'}{q}} \Phi(x)^{q'} dx \lesssim \int_0^1 V_\varepsilon(x)^{-\frac{q'}{q}} l x^{(\alpha+\frac{1}{2})q'} dx,$$

which is finite by (i) in the proposition (choosing ε sufficiently small). On the other hand,

$$(6.12) \quad \int_1^\infty v_{1,\varepsilon}(x)^{-\frac{q'}{q}} \Phi(x)^{q'} dx \lesssim \int_1^\infty V_\varepsilon(x)^{-\frac{q'}{q}} e^{-q'(1-\frac{p}{q})\frac{x^2}{2}} (1+x)^{2\nu p q'/q} dx,$$

and since $W \in D_p^{\text{exp}}(a)$ for all $a > 0$, we can apply part (ii) of Proposition 3.2 (for a sufficiently small ε) to conclude that this is also a finite quantity. \square

Remark 6.2. Alternative expression for the second weight. A slight modification of the above construction suggests to define a new weight by

$$(6.13) \quad v_\varepsilon^{\Phi,w}(x) := \min \left\{ \Phi(x)^p \left[\mathcal{M}^{\text{loc}}(w^{-\frac{p'}{p}} \Phi^{p'})(x) \right]^{-\frac{p}{p'}} \Upsilon_\varepsilon(x), \frac{lx^{p-1}}{[\log(e/lx)]^2} \frac{\Phi(x)^p}{(1+x)^{N_0}} \right\}$$

with

$$\Upsilon_\varepsilon(x) = \frac{lx^{\varepsilon N_1}}{(1+x)^{N_2}} \rho_\varepsilon \left(\left[\mathcal{M}^{\text{loc}}(w^{-\frac{p'}{p}} \Phi^{p'})(x) \right]^{-\frac{p}{p'}} \right).$$

If N_0, N_1, N_2 are sufficiently large, then similar arguments as above lead to the boundedness of $P_{t_0}^* : L^p(w) \rightarrow L^p(v_\varepsilon^{\Phi, w})$ for all $\varepsilon > 0$, and give also the property that $v_\varepsilon^{\Phi, w} \in D_q(\Phi)$ if ε is sufficiently small. We omit the details. The expression in (6.13) has the advantage of remaining valid for the other Laguerre systems in Table 1 (with the corresponding Φ functions).

7. TRANSFERENCE TO OTHER LAGUERRE TYPE SYSTEMS

In this section we show how to transfer the results already proved for the system $\{\varphi_n^\alpha\}$ and the operator L to the other Laguerre systems and operators in Table 1. The procedure is completely general, as one can infer already from the first two cases.

7.1. Results for the system ψ_n^α . The starting point is the identity defining ψ_n^α , namely

$$(7.1) \quad \psi_n^\alpha(y) = a(y) \varphi_n^\alpha(y), \quad \text{with } a(y) = y^{-\alpha-\frac{1}{2}}.$$

Clearly, φ_n^α is an eigenvector of L if and only if ψ_n^α is an eigenvector of the operator

$$f \mapsto \Lambda f(x) = a(x) L[a^{-1}f](x)$$

(with the same eigenvalue $\lambda_n = 4n + 2(\alpha + 1 + \mu)$). An elementary computation shows that the differential operator Λ obtained in this fashion is exactly the one listed in Table 1. Remark also that $\{\psi_n^\alpha\}$ becomes an orthonormal basis in L^2 with the measure $a^{-2}(y)dy = y^{2\alpha+1}dy$.

The identity in (7.1) leads to a pointwise relation of the corresponding heat kernels

$$e^{-t\Lambda}(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n^\alpha(x) \psi_n^\alpha(y) = a(x) a(y) e^{-tL}(x, y),$$

and by the subordination formula, also of the corresponding Poisson kernels

$$P_t^\Lambda(x, y) = a(x) a(y) P_t^L(x, y).$$

In particular,

$$(7.2) \quad P_t^\Lambda f(x) = \int_{\mathbb{R}_+} P_t^\Lambda(x, y) f(y) a^{-2}(y) dy = a(x) P_t^L[a^{-1}f](x).$$

From this relation it is clear that Theorem 1.1 becomes true for the operator Λ with

$$\Phi^\Lambda(y) = a(y)^{-1} \Phi^L(y),$$

as listed in Table 1. From (7.2) it also follows that $P_{t_0}^{*, \Lambda}$ maps $L^p(w) \rightarrow L^p(v)$ if and only if $P_{t_0}^{*, L}$ maps $L^p(a^p w) \rightarrow L^p(a^p v)$, and hence the necessary and sufficient condition becomes

$$a^p w \in D_p(\Phi^L) \iff \|a^{-1} w^{-\frac{1}{p}} \Phi^L\|_{p'} < \infty \iff w \in D_p(\Phi^\Lambda),$$

as was claimed in Theorem 1.2. For the assertions about the weight v one may argue directly as follows. Observe from (7.2) and Corollary 5.3 that we can write

$$P_{t_0}^{*, \Lambda} f(x) \lesssim C_1(x) a(x) \mathcal{M}_M^{\text{loc}}(f a^{-1} e^{-\frac{y^2}{2}})(x) + C_2(x) a(x) \int_{\mathbb{R}_+} |f| \Phi^\Lambda,$$

with a cancellation in the first term due to $a(x)a(y)^{-1} \approx 1$ when $\frac{x}{2} < y < Mx$. At this point we can apply the same arguments as in §6.2. Namely, we construct $v = \min\{v_{1,\varepsilon}, v_2\}$ with the same choice of $v_{1,\varepsilon}$, and with v_2 in (6.5) now replaced by

$$v_2(x) := \frac{lx^{(2\alpha+2)p-1}}{[\log(e/lx)]^2} \frac{e^{-\frac{p}{2}x^2}}{(1+x)^N}.$$

The same proof will give that, for any $q > p$, there is a sufficiently small ε so that $v \in D_q(\Phi^\Lambda)$ (the only difference being that, locally, this condition now becomes $v \in D_q^0(2\alpha+1)$). We remark that this part will work as well with the choice

$$v(x) = a^{-p}(x) v^{\Phi^L, a^p w}(x) = v^{\Phi^\Lambda, w}(x),$$

as defined in (6.13).

7.2. Results for the system \mathfrak{L}_n^α . Consider the following isometry of $L^2(\mathbb{R}_+, dy)$

$$f \longmapsto Af(x) = \sqrt{2x} f(x^2).$$

The systems \mathfrak{L}_n^α and φ_n^α are related by $\varphi_n^\alpha = A\mathfrak{L}_n^\alpha$, or equivalently

$$(7.3) \quad \mathfrak{L}_n^\alpha(y) = [A^{-1}\varphi_n^\alpha](y) = (4y)^{-\frac{1}{4}} \varphi_n^\alpha(\sqrt{y}).$$

In particular, \mathfrak{L}_n^α is an eigenvector of the operator

$$\mathfrak{L} = \frac{1}{4} A^{-1} \circ L \circ A,$$

this time with eigenvalue $\lambda_n/4 = n + (\alpha + 1 + \mu)/2$. The factor $\frac{1}{4}$ has been added so that \mathfrak{L} coincides with the operator listed in Table 1.

The heat kernels are now related by

$$e^{-t\mathfrak{L}}(x, y) = \sum_{n=0}^{\infty} e^{-t\lambda_n/4} \mathfrak{L}_n^\alpha(x) \mathfrak{L}_n^\alpha(y) = \frac{1}{2(xy)^{\frac{1}{4}}} e^{-\frac{t}{4}L}(\sqrt{x}, \sqrt{y}),$$

and therefore, a substitution in the subordinated integral in (5.5) gives

$$P_t^\mathfrak{L}(x, y) = (16xy)^{-\frac{1}{4}} P_{t/2}^L(\sqrt{x}, \sqrt{y}).$$

Thus, we obtain the formula

$$(7.4) \quad P_t^\mathfrak{L}f(x) = \int_{\mathbb{R}_+} P_t^\mathfrak{L}(x, y) f(y) dy = (4x)^{-\frac{1}{4}} P_{t/2}^L[Af](\sqrt{x}).$$

From this relation one easily deduces Theorem 1.1 for the operator \mathfrak{L} , provided that

$$(7.5) \quad \Phi^\mathfrak{L}(y) = [A^{-1}\Phi^L](y),$$

which is the function asserted in Table 1 (modulo constants).

To establish the second theorem, first observe from (7.4) and Corollary 5.3 that

$$P_{t_0}^{*, \mathfrak{L}}f(x) \lesssim \frac{C_1(\sqrt{x})}{x^{1/4}} \mathcal{M}_M^{\text{loc}}(\sqrt{y}f(y^2)e^{-\frac{y^2}{2}})(\sqrt{x}) + \frac{C_2(\sqrt{x})}{x^{1/4}} \int_{\mathbb{R}_+} \sqrt{2y}|f(y^2)|\Phi^L(y).$$

We claim that this inequality can be rewritten as

$$(7.6) \quad P_{t_0}^{*, \mathfrak{L}} f(x) \lesssim C_1(\sqrt{x}) \mathcal{M}_{M^2}^{\text{loc}}(f e^{-\frac{y}{2}})(x) + \frac{C_2(\sqrt{x})}{x^{1/4}} \int_{\mathbb{R}_+} |f(u)| \Phi^{\mathfrak{L}}(u).$$

The expression for the second term is clear from (7.5) (after a change of variables $y^2 = u$). To handle the first term, notice that the local region now becomes $\frac{\sqrt{x}}{2} < y < M\sqrt{x}$, which in particular gives $x^{-\frac{1}{4}}\sqrt{y} \approx 1$. We also need the following lemma for the maximal function.

Lemma 7.1. *For all $g \geq 0$ and $x \in \mathbb{R}_+$,*

$$\mathcal{M}\left(g(y^2) \chi_{\{\frac{\sqrt{x}}{2} < y < M\sqrt{x}\}}\right)(\sqrt{x}) \lesssim \mathcal{M}\left(g(u) \chi_{\{\frac{x}{4} < u < M^2 x\}}\right)(x).$$

PROOF: This follows essentially from the change of variables $y^2 = u$,

$$\begin{aligned} LHS &\leq \sup_{r>0} \frac{1}{r} \int_{|y-\sqrt{x}|<r} g(y^2) \chi_{\{\frac{x}{4} < y^2 < M^2 x\}} dy \\ &= \sup_{r>0} \frac{1}{r} \int_{|\sqrt{u}-\sqrt{x}|<r} g(u) \chi_{\{\frac{x}{4} < u < M^2 x\}} \frac{du}{2\sqrt{u}}. \end{aligned}$$

In this local range we have $\sqrt{u} \approx \sqrt{x}$, so may take the denominator outside the integral. The local behavior also implies that

$$|\sqrt{u} - \sqrt{x}| = \left| \int_x^u \frac{ds}{2\sqrt{s}} \right| \approx \frac{|u - x|}{\sqrt{x}}.$$

Therefore, we conclude that

$$LHS \lesssim \sup_{r>0} \frac{1}{r\sqrt{x}} \int_{|u-x|<r\sqrt{x}} g(u) \chi_{\{\frac{x}{4} < u < M^2 x\}} du \lesssim RHS. \quad \square$$

Remark 7.2. A small variation of this proof shows that $x \in \mathcal{L}_f$ implies $\sqrt{x} \in \mathcal{L}_{Af}$, so in view of (6.1), the pointwise convergence in (ii) of Theorem 1.1 (for the operator \mathfrak{L}) actually holds at every Lebesgue point x of f .

We have thus shown (7.6). From here one proves Theorem 1.2 (for the operator \mathfrak{L}) arguing once again as in §6.2. Remark that, in view of the new constants C_1 and C_2 in (7.6), the weight $v = \min\{v_{1,\varepsilon}, v_2\}$ must be defined with

$$v_{1,\varepsilon}(x) = \frac{e^{-\frac{px}{2}}}{(1+x)^{p\nu}} \mathcal{V}(x) \rho_\varepsilon(\mathcal{V}(x)) \quad \text{and} \quad v_2(x) = \frac{lx^{(\frac{\alpha}{2}+1)p-1}}{[\log(e/lx)]^2} \frac{e^{-\frac{px}{2}}}{(1+x)^N},$$

where $\mathcal{V}(x) = \left[\mathcal{M}_{M^2}^{\text{loc}}(w^{\frac{1}{1-p}} e^{-\frac{p'x}{2}})(x) \right]^{1-p}$ and $N > 1 + \frac{p}{2}(|\mu + \frac{1}{2}| - \frac{1}{2})$. Then, the same proof as before gives that $P_{t_0}^* : L^p(w) \rightarrow L^p(v)$ if $w \in D_p(\Phi^{\mathfrak{L}})$. One can also establish (with a few obvious modifications) that for every given $q > p$, there is a sufficiently small ε so that $v \in D_q(\Phi^{\mathfrak{L}})$. Once again, we may also replace this weight by $v^{\Phi^{\mathfrak{L}}, w}(x)$, as defined in (6.13).

7.3. Results for the system ℓ_n^α . Remember that these functions satisfy

$$(7.7) \quad \ell_n^\alpha(y) = a(y) \mathfrak{L}_n^\alpha(y), \quad \text{with } a(y) = y^{-\frac{\alpha}{2}}.$$

Thus, they are eigenvectors of the differential operator

$$f \mapsto \mathcal{L}f(x) = a(x) \mathfrak{L}[a^{-1}f](x)$$

(with the same eigenvalues as \mathfrak{L}_n^α) and constitute an orthonormal system in $L^2(a(y)^{-2}dy)$. One then derives Theorems 1.1 and 1.2 for the operator \mathcal{L} , from the known results about \mathfrak{L} , by repeating exactly the same arguments that we gave in §7.1. We leave the details to the reader.

7.4. Results for the Laguerre polynomials L_n^α . This system and the corresponding operator \mathbb{L} in (1.1) are the ones considered in the statements of §1, so we shall give a few more details here. First of all, recall that L_n^α and \mathfrak{L}_n^α are linked by

$$(7.8) \quad L_n^\alpha(y) = a(y) \mathfrak{L}_n^\alpha(y), \quad \text{with } a(y) = y^{-\frac{\alpha}{2}} e^{y/2}.$$

Thus, the functions L_n^α are orthonormal in $L^2(a(y)^{-2}dy) = L^2(y^\alpha e^{-y}dy)$, and are also eigenvectors of the differential operator

$$f \mapsto \mathbb{L}f(x) = a(x) \mathfrak{L}[a^{-1}f](x),$$

with the same eigenvalues as \mathfrak{L}_n^α , namely $n + (\alpha + 1 + \mu)/2$. We remark that \mathbb{L} coincides with the operator defined in (1.1) when we set $m = (\alpha + 1 + \mu)/2$. Thus, the heat and Poisson kernels of these two operators are related by

$$e^{-t\mathbb{L}}(x, y) = \sum_{n=0}^{\infty} e^{-(n+m)t} L_n^\alpha(x) L_n^\alpha(y) = a(x) a(y) e^{-t\mathfrak{L}}(x, y),$$

and

$$P_t^{\mathbb{L}}(x, y) = a(x) a(y) P_t^{\mathfrak{L}}(x, y).$$

This implies the identity

$$(7.9) \quad P_t^{\mathbb{L}}f(x) = \int_{\mathbb{R}_+} P_t^{\mathbb{L}}(x, y) f(y) y^\alpha e^{-y} dy = a(x) P_t^{\mathfrak{L}}[f y^{\frac{\alpha}{2}} e^{-\frac{y}{2}}](x),$$

from which one deduces the validity of Theorem 1.1 for the operator \mathbb{L} , provided

$$\Phi^{\mathbb{L}}(x) = y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} \Phi^{\mathfrak{L}}(y) = \frac{y^\alpha e^{-y}}{(1+y)^{\frac{1+\alpha+\mu}{2}} [\log(e+x)]^{1+\nu}}.$$

Note that this coincides with the function in (1.4) since we have set $m = (\alpha + 1 + \mu)/2$. Moreover, (7.9) combined with (7.6) implies the estimate

$$(7.10) \quad P_{t_0}^{*,\mathbb{L}}f(x) \lesssim C_1^{\mathbb{L}}(x) \mathcal{M}_{M^2}^{\text{loc}}(f e^{-y})(x) + C_2^{\mathbb{L}}(x) \int_{\mathbb{R}_+} |f(u)| \Phi^{\mathbb{L}}(u) = I(x) + II(x),$$

with the new constants

$$C_1^{\mathbb{L}}(x) = (1+x)^\nu e^x \quad \text{and} \quad C_2^{\mathbb{L}}(x) = (1+x)^{(|\mu+\frac{1}{2}|-\alpha-\frac{1}{2})/2} e^x / l x^{\alpha+1}.$$

We now apply the same arguments as in §6.2 to show that, for a suitable weight v , we have $\|P_{t_0}^{*,\mathbb{L}} f\|_{L^p(v)} \lesssim \|f\|_{L^p(w)}$, under the assumption $w \in D_p(\Phi^{\mathbb{L}})$. Indeed, to control the second term $II(x)$ we choose a weight v_2 such that $C_2^{\mathbb{L}} \in L^p(v_2)$, namely

$$v_2(x) = \frac{lx^{(\alpha+1)p-1}}{[\log(e/lx)]^2} \frac{e^{-px}}{(1+x)^N},$$

with say $N > 1 + p(m + |\alpha + \frac{1}{2}|)$. It is not difficult to verify that $v_2 \in D_q(\Phi^{\mathbb{L}})$ for all $q > p$. To control the first term we set

$$v_{1,\varepsilon}(x) = \frac{e^{-px}}{(1+x)^{p\nu}} \mathcal{V}(x) \rho_\varepsilon(\mathcal{V}(x)) \text{ with } \mathcal{V}(x) = \left[\mathcal{M}_{M^2}^{\text{loc}}(w^{\frac{1}{1-p}} e^{-p'x})(x) \right]^{1-p}.$$

That is, if $W(x) = w(x)e^{px}$, then $v_{1,\varepsilon}(x) = (1+x)^{-p\nu} e^{-px} V_\varepsilon(x)$ with the notation in (3.1). So we may quote Theorem 3.1 to obtain

$$\|I\|_{L^p(v_{1,\varepsilon})} \lesssim \left[\int_{\mathbb{R}_+} |\mathcal{M}_{M^2}^{\text{loc}}(fe^{-y})(x)|^p V_\varepsilon(x) dx \right]^{\frac{1}{p}} \lesssim \|fe^{-y}\|_{L^p(W)} = \|f\|_{L^p(w)}.$$

Again, it is not difficult to verify that for a sufficiently small ε one has $v_{1,\varepsilon} \in D_q(\Phi^{\mathbb{L}})$, arguing as in the last part[§] of §6.2. Thus, Theorem 1.2 holds with $v = \min\{v_{1,\varepsilon}, v_2\}$. Alternatively, with the notation in (6.13), one may as well choose the weight $v^{\Phi^{\mathbb{L}},w}(x)$.

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[§]With the quadratic exponentials $e^{x^2/2}$ in (6.7) and (6.12) replaced by linear exponentials e^x .

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